Finitary, Causal and Quantal Vacuum Einstein Gravity

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Abstract

We continue recent work [74, 75] and formulate the gravitational vacuum Einstein equations over a locally finite spacetime by using the basic axiomatics, techniques, ideas and working philosophy of Abstract Differential Geometry. The main kinematical structure involved, originally introduced and explored in [74], is a curved principal finitary spacetime sheaf of incidence algebras, which have been interpreted as quantum causal sets, together with a non-trivial locally finite spin-Loretzian connection on it which lays the structural foundation for the formulation of a covariant dynamics of quantum causality in terms of sheaf morphisms. Our scheme is innately algebraic and it supports a categorical version of the principle of general covariance that is manifestly independent of a background $\mathcal{C}^\infty$-smooth spacetime manifold M. Thus, we entertain the possibility of developing a 'fully covariant' path integral-type of quantum dynamical scenario for these connections that avoids *ab initio* various problems that such a dynamics encounters in other current quantization schemes for gravity—either canonical (Hamiltonian), or covariant (Lagrangian)—involving an external, base differential spacetime manifold, namely, the choice of a diffeomorphism-invariant measure on the moduli space of gauge-equivalent (self-dual) gravitational spin-Lorentzian connections and the (Hilbert space) inner product that could in principle be constructed relative to that measure in the quantum theory—the so-called 'inner product problem', as well as the 'problem of time' that also involves the $\mathrm{Diff}(M)$ 'structure group' of the classical $\mathcal{C}^\infty$-smooth spacetime continuum of general relativity. Hence, by using the inherently algebraico-sheaf-theoretic and calculus-free ideas of Abstract Differential Geometry, we are able to draw preliminary, albeit suggestive, connections between certain non-perturbative (canonical or covariant) approaches to quantum general relativity (*eg*, Ashtekar's new variables and the loop formalism that has been developed along with them) and Sorkin *et al.*'s causal set program—as it were, we 'noncommutatively algebraize', 'differential geometrize' and, as a result, dynamically vary causal sets. At the end, we anticipate various consequences that such a scenario for a locally finite, causal and quantal vacuum Einstein gravity might have for the obstinate from the viewpoint of the smooth continuum problem of $\mathcal{C}^\infty$-smooth spacetime singularities, thus we prepare the ground for a forthcoming paper [76].

PACS numbers: 04.60.-m, 04.20.Gz, 04.20.-q

Key words: quantum gravity, causal sets, differential incidence algebras of locally finite partially ordered sets, abstract differential geometry, sheaf theory, sheaf cohomology, category theory

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"*...the theory that space is continuous is wrong, because we get...infinities* [viz. 'singularities'] *and other similar difficulties* ...[while] *the simple ideas of geometry, extended down to infinitely small, are wrong...*" [39]

....

"*...at the Planck-length scale, classical differential geometry is simply incompatible with quantum theory...*[so that] *one will not be able to use differential geometry in the true quantum-gravity theory...*" [52]

1 Prologue cum Physical Motivation

In the last century, the path that we have followed to unite quantum mechanics with general relativity into a coherent, both technically and conceptually, quantum theory of gravity has been a long and arduous one, full of unexpected twists and turns, surprising detours, branchings and loops—even disheartening setbacks and impasses, as well as hopes, disappointments or even disillusionments at times. Certainly though, the whole enterprize has been supported and nurtured by impressive technical ingenuity, and creative imagination coming from physicists and mathematicians alike. All in all, it has been a trip of adventure, discovery and intellectual reward for all who have been privileged to be involved in this formidable quest. Arguably then, the attempt to arrive at a conceptually sound and 'calculationally' finite quantum gravity must be regarded and hailed as one of the most challenging and inspired endeavors in theoretical physics research that must be carried over and be zestfully continued in the new millenium.

Admittedly, however, a cogent theoretical scenario for quantum gravity has proved to be stubbornly elusive not least because there is no unanimous agreement about what ought to qualify as the 'proper' approach to a quantum theoresis of spacetime and gravity. Generally speaking, most of the approaches fall into the following three categories:[1]

1. *'General relativity conservative'*: The general aim of the approaches falling into this category is to quantize classical gravity somehow. Thus, the mathematical theory on which general relativity—in fact, any field theory whether classical or quantum—is based, namely, the differential geometry of $\mathcal{C}^\infty$-manifolds (*ie*, the usual differential calculus on manifolds), is essentially retained[2] and it is used to treat the gravitational field quantum field-theoretically. Both the non-perturbative canonical and covariant (*ie*, path integral or 'action-weighed sum-over-histories') approaches to 'quantum general relativity', topological quantum field theories, as well as, to a large extent, higher dimensional (or extended objects') theories like (super)string and membrane schemes arguably belong to this category.

[1]These categories should by no means be regarded as being mutually exclusive or exhaustive, and they certainly reflect only these authors' subjective criteria and personal perspective on the general characteristics of various approaches to quantum gravity. This coarse classification will be useful for the informal description of our finitary and causal approach to Lorentzian vacuum quantum gravity to be discussed shortly.

[2]That is, in general relativity spacetime is modelled after a $\mathcal{C}^\infty$-smooth manifold. Purely mathematically speaking, approaches in this category could also be called '$\mathcal{C}^\infty$-smoothness or differential manifold conservative'.

2. *'Quantum mechanics conservative'*: The general spirit here is to start from general quantum principles such as algebraic operationality, noncommutativity and finitism ('discreteness') about the structure of spacetime and its dynamics, and then try to derive somehow general relativistic attributes, as it were, from within the quantum framework. Such approaches assume up-front that quantum theory is primary and fundamental, while the classical geometrical smooth spacetime continuum and its dynamics secondary and derivative (emergent) from the deeper quantum dynamical realm. For instance, Connes' noncommutative geometry [24, 57] and, perhaps more notably, Finkelstein's quantum relativity [45, 106][3] may be classified here.

3. *'Independent'*: Approaches in this category assume neither quantum mechanics nor general relativity as a fundamental, 'fixed' background theory relative to which the other must be modified to suit. Rather, they start independently from principles that are neither quantum mechanical nor general relativistic *per se*, and proceed to construct a theory and a suitable mathematical formalism to accompany it that later may be interpreted as a coherent amalgamation (or perhaps even extension) of both. It is inevitable with such 'iconoclastic' schemes that in the end both general relativity and quantum mechanics may appear to be modified to some extent. One could assign to this category Penrose's combinatorial spin-networks [83, 101] and its current relativistic spin-foam descendants [13, 15, 84], Regge's homological spacetime triangulations and simplicial gravity [94], as well as Sorkin *et al.*'s causal sets [18, 115, 116, 95, 117].

It goes without saying that this is no place for us to review in any detail the approaches mentioned above.[4] Rather, we wish to continue a finitary, causal and quantal sheaf-theoretic approach to spacetime and vacuum Lorentzian gravity that we have already started to develop in [74, 75]. This approach, as we will argue subsequently, combines characteristics from all three categories above and, in particular, the mathematical backbone which supports it, *Abstract Differential Geometry* (ADG) [67, 68, 70, 71, 73], was originally conceived in order to evade the $\mathcal{C}^\infty$-smooth spacetime manifold M (and consequently its diffeomorphism group $\mathrm{Diff}(M)$) underlying (and creating numerous problems for) the various approaches in 1. For, it must be emphasized up-front, *ADG is an axiomatic formulation of differential geometry which does not use any $\mathcal{C}^\infty$-notion from the usual differential calculus—the classical differential geometry of smooth manifolds.*

To summarize briefly what we have already accomplished in this direction,[5] in [74] we com-

[3] In fact, Finkelstein maintains that "*all is quantum. Anything that appears to be classical has not yet been resolved into its quantum elements*" (David Finkelstein in private communication).

[4] For reviews of and different perspectives on the main approaches to quantum gravity, the reader is referred to [54, 4, 99]. In the last, most recent reference, one notices a similar partition of the various approaches to quantum gravity into three classes called *covariant*, *canonical* and *sum-over-histories*. Then one realizes that presently we assigned all these three classes to category 1, since our general classification criterion is which approaches, like general relativity, more-or-less preserve a $\mathcal{C}^\infty$-smooth base spacetime manifold hence use the methods of the usual differential geometry on it, and which do not. Also, by 'covariant' we do not mean what Rovelli does. 'Covariant' for us is synonymous to 'action-weighed sum-over-histories' or 'path integral'. Undoubtedly, there is arbitrariness and subjectivity in such denominations, so that the boundaries of those distinctions are rather fuzzy.

[5] For a recent, concise review of our work so far on this sheaf-theoretic approach to discrete Lorentzian quantum

bined ideas from the second author's work on *finitary spacetime sheaves*[6] (finsheaves) [87] and on an algebraic quantization scenario for Sorkin's causal sets (causets) [86] with the first author's ADG [67, 68], and we arrived at a locally finite, causal and quantal version of the kinematical structure of Lorentzian gravity. The latter pertains to the definition of a curved principal finsheaf $\vec{\mathcal{P}}_i^{\uparrow}$ of incidence Rota algebras modelling *quantum causal sets* (qausets) [86], having for structure group a locally finite version of the continuous orthochronous Lorentz group $SO(1,3)^{\uparrow}$ of local symmetries (isometries) of general relativity, together with a non-trivial (*ie*, non-flat) locally finite $so(1,3)_i^{\uparrow} \simeq sl(2,\mathbb{C})_i$-valued spin-Lorentzian connection $\vec{\mathcal{D}}_i$[7] which represents the localization or gauging and concomitant dynamical variability of the qausets in the sheaf due to a finitary, causal and quantal version of Lorentzian gravity in the absence of matter (*ie*, vacuum Einstein gravity). We also gave the following quantum particle interpretation to this reticular scheme: a so-called *causon*—the elementary particle of the field of dynamical quantum causality represented by $\vec{\mathcal{A}}_i$—was envisioned to dynamically propagate in the reticular curved spacetime vacuum represented by the finsheaf of qausets under the influence of finitary Lorentzian (vacuum) quantum gravity.

In the sequel [75], by using the universal constructions and the powerful sheaf-cohomological tools of ADG together with the rich differential structure with which the incidence algebras modelling qausets are equipped [92, 86, 93, 130, 128], we showed how basic differential geometric ideas and results usually thought of as being vitally dependent on $\mathcal{C}^\infty$-smooth manifolds for their realization, as for example the standard Čech-de Rham cohomology, carry through virtually unaltered to the finitary regime of the curved finsheaves of qausets. For instance, we gave finitary versions of important $\mathcal{C}^\infty$-theorems such as de Rham's, Weil's integrality and the Chern-Weil theorem and, based on certain robust results from the application of ADG to the theory of geometric (pre)quantization [68, 69, 72], we carried out a sheaf-cohomological classification of the associated line sheaves bearing the finitary spin-Lorentzian $\vec{\mathcal{A}}_i$s whose quanta were referred to as causons above—the elementary (bosonic) particles carrying the dynamical field of quantum causality whose (local) states correspond precisely to (local) sections of those line sheaves. By this virtually complete transcription of the basic $\mathcal{C}^\infty$-constructions, concepts and results to the locally finite and quantal realm of the curved finsheaves of qausets, we highlighted that for their formulation the classical smooth background spacetime continuum is essentially of no contributing value. Moreover, we argued that since the $\mathcal{C}^\infty$-smooth spacetime manifold can be regarded as the main culprit for the singularities that plague general relativity as well as for the weaker but still troublesome infinities that assail the flat quantum field theories of matter, its evasion—especially by the finitistic-algebraic means that we employed—should be most welcome for the formulation

gravity, as well as on its possible topos-theoretic extension, the reader is referred to [90]. In particular, the topos-theoretic viewpoint is currently being elaborated in [91].

[6]Throughout this paper, the epithets 'finitary' and 'locally finite' will be used interchangeably.

[7]From [74] we note that only the gauge potential $\vec{\mathcal{A}}_i$ part of the reticular $\vec{\mathcal{D}}_i = \vec{\partial}_i + \vec{\mathcal{A}}_i$ is spin-Lorentzian proper (*ie*, discrete $so(1,3)_i^{\uparrow} \simeq sl(2,\mathbb{C})_i$-valued), but here too we will abuse terminology and refer to either $\vec{\mathcal{D}}_i$ or its part $\vec{\mathcal{A}}_i$ as 'the spin-Lorentzian connection'. (The reader should also note that the arrows over the various symbols will be justified in the sequel in view of the causal interpretation that our incidence algebra finsheaves have; while, the subscript 'i' is the so-called 'finitarity', 'resolution', or 'localization index' [87, 74, 75], which we will also explain in the sequel.)

of a 'calculationally' and, in a sense to be explained later, 'inherently finite' and 'fully covariant' quantum theory of gravity.

With respect to the aforementioned three categories of approaches to quantum gravity, our scheme certainly has attributes of 2 as it employs finite dimensional non-abelian incidence algebras to model (dynamically variable) qausets in the stalks of the relevant finsheaves, which qausets have a rather natural quantum-theoretic (because algebraico-operational) physical interpretation [92, 86, 74, 93]. It also has traits of category 3 since the incidence algebras are, by definition, of combinatorial and 'directed simplicial' homological character and, in particular, Sorkin's causet theory was in effect its principal physical motivation [86, 75]. Finally, regarding category 1, the purely mathematical, ADG-based aspect of our approach was originally motivated by a need to show that *all the 'intrinsic' differential mechanism of the usual calculus on manifolds is independent of $\mathcal{C}^\infty$-smoothness*, in fact, *of any notion of 'space' supporting the usual differential geometric concepts and constructions*,[8] thus entirely avoid, or better, manage to integrate or 'absorb' into the (now generalized) abstract differential geometry, the 'anomalies' (*ie*, the singularities and other 'infinity-related pathologies') that plague the classical $\mathcal{C}^\infty$-smooth continuum case [67, 68, 73]. Arguably then, our approach is an amalgamation of elements from 1–3.

Let us now move on to specifics. In the present paper we continue our work in [74, 75] and formulate the dynamical vacuum Einstein equations in $\vec{\mathcal{P}}_i^\uparrow$. On the one hand, this extends our work on the kinematics of a finitary and causal scheme for Lorentzian quantum gravity developed in [74] as it provides a suitable dynamics for it, and on the other, it may be regarded as another concrete physical application of ADG to the locally finite, causal and quantum regime, and all this *in spite of the $\mathcal{C}^\infty$-smooth spacetime manifold*, in accord with the spirit of [75]. Our work here is the second physical application of ADG to vacuum Einstein gravity, the first having already involved the successful formulation of the vacuum Einstein equations over spaces with singularities concentrated on arbitrary closed nowhere dense sets—arguably, *the* most singular spaces when viewed from the featureless $\mathcal{C}^\infty$-smooth spacetime manifold perspective [70, 77, 78, 73, 97].

The paper is organized as follows: in the following section we recall the basic ideas about connections in ADG focusing our attention mainly on Yang-Mills (Y-M) and Lorentzian connections on finite dimensional vector sheaves, on principal sheaves (whose associated sheaves are the aforementioned vector sheaves), their curvatures, symmetries and (Bianchi) identities, as well as

[8]Thus, as we will time and again stress in the sequel, with the development of ADG we have come to realize that the main operative role of the $\mathcal{C}^\infty$-smooth manifold is to provide us with *a* convenient (and quite successful in various applications to both classical and quantum physics!), *but by no means unique*, differential mechanism, namely, that accommodated by the algebra $\mathcal{C}^\infty(M)$ of infinitely differentiable functions 'coordinatizing' the (points of the) differential manifold M. However, the latter algebra's pathologies in the form of singularities made us ponder on the question whether the differential mechanism itself is 'innate' to $\mathcal{C}^\infty(M)$ and the manifold supporting these 'generalized arithmetics' (this term is borrowed straight from ADG). As alluded to above, ADG's answer to the latter is an emphatic '*No*!' [67, 68, 73]. For example, one can do differential geometry over very (in fact, *the most*!) singular from the point of view of the $\mathcal{C}^\infty$-smooth M spaces and their 'arithmetic algebras', such as Rosinger's non-linear distributions—the so-called differential algebras of generalized functions) [77, 78, 97]. As a matter of fact, the last two papers, together with the duet [74, 75], are examples of two successful applications of ADG proving its main point, that: "*differentiability is independent of $\mathcal{C}^\infty$-smoothness*" (see slogan 2 at the end of [75]).

the affine spaces that they constitute. In section 3 we discuss the connection-based picture of gravity—the way in which general relativity may be thought of as a Y-M-type of gauge theory in the manner of ADG [71]. Based mainly on [70], we present vacuum Einstein gravity *à la* ADG and explore the relevant gravitational moduli spaces of spin-Lorentzian connections. In section 4 we remind the reader of some basic kinematical features of our curved principal finsheaves of qausets from [74, 75] and, in particular, based on recent results of Papatriantafillou and Vassiliou [81, 82, 124, 125, 126], we describe in a categorical way inverse (projective) and direct (inductive) limits of such principal finsheaves and their reticular connections. We also comment on the use of the real ($\mathbb{R}$) and complex ($\mathbb{C}$) number fields in our manifold-free, combinatory-algebraic theory, and compare it with some recent critical remarks of Isham [55] about the *a priori* assumption—one that is essentially based on the classical manifold model of spacetime—of the $\mathbb{R}$ and $\mathbb{C}$ continua in conventional quantum theory *vis-à-vis* its application to quantum gravity. Section 5 is the focal area of this paper as it presents a locally finite, causal and quantal version of the vacuum Einstein equations for Lorentzian gravity. The idea is also entertained of developing a possible covariant quantization scheme for finitary Lorentzian gravity involving a path integral-type of functional over the moduli space $\vec{\mathcal{A}}_i/\mathcal{G}_i$ of all reticular gauge-equivalent spin-Lorentzian connections $\vec{\mathcal{A}}_i$. Based on the 'innate' finiteness of our model, we discuss how such a scenario may on the one hand avoid *ab initio* the choice of measure for $\vec{\mathcal{A}}_i$ that troubles the continuum functional integrals over the infinite dimensional, non-linear and with a 'complicated' topology moduli space $\mathcal{A}^{(+)}_{\infty}/\mathcal{G}$ of smooth, (self-dual) Lorentzian connections in the standard covariant approach to the quantization of (self-dual) Lorentzian gravity, and on the other, how our up-front avoiding of $\mathrm{Diff}(M)$ may cut the 'Gordian knot' that the problems of time and of the inner product in the Hilbert space of physical states present to the non-perturbative canonical approach to quantum gravity based on Ashtekar's new variables and the holonomy (Wilson loop) formalism associated with them. Ultimately, all this points to the fact that *our theory is genuinely $\mathcal{C}^{\infty}$-smooth spacetime background independent* and, perhaps more importantly, *regardless of the perennial debate whether classical (vacuum) gravity should be quantized covariantly or canonically.* This makes us ask—in fact, altogether doubt—whether quantizing classical spacetime and gravity by using the constructions and techniques of the usual differential geometry of smooth manifolds is the 'right' approach to quantum spacetime and gravity, thus align ourselves more with the categories 2 and 3 above, and less with 1. As a matter of fact, and in contradistinction to the 'iconoclastic' approaches in category 3 (most notably, in contrast to the theory of causal sets), in developing our entirely algebraico-sheaf-theoretic approach to finitary Lorentzian quantum gravity based on ADG, we have come to question altogether whether the notion of (an inert geometrical background) 'spacetime'—whether it is modelled after a continuous or a discrete base space—makes any physical sense in the ever dynamically fluctuating quantum deep where the vacuum is 'filled' solely by (the dynamics of) causons and where there is no 'ambient' or surrounding spacetime that actively participates into or influences in any way that dynamics.[9] We thus infer that both

[9]Of course, we will see that there is a base topological 'localization space'—a stage on which we solder our algebraic structures, but this space is of an ether-like character, a surrogate scaffolding of no physical significance whatsoever as it does not actively engage into the quantum dynamics of the causons—the quanta of the field $\vec{\mathcal{A}}_i$ of quantum causality that is localized (gauged) and dynamically propagates on 'it'.

our finitary vacuum Einstein equations for the causon and the path integral-like quantum dynamics of our reticular (self-dual) spin-Lorentzian connections $\vec{\mathcal{A}}_i^{(+)}$ is 'genuinely', or better, 'fully' covariant since they both concern directly and solely the objects (the quanta of causality, *ie*, the dynamical connections $\vec{\mathcal{A}}_i$) that live on that base 'space(time)', and not at all that external, passive and dynamically inert 'space(time) arena' itself. We also make comments on geometric (pre)quantization [68, 69, 72] in the light of our application here of ADG to finitary and causal Lorentzian gravity [75] and we stress that our scheme may be perceived as being, in a strong sense, 'already' or 'inherently' quantum, meaning that it is in no need of the (formal) process of quantization of the corresponding classical theory (here, general relativity on a $\mathcal{C}^\infty$-smooth spacetime manifold). This seems to support further our doubts about the quantization of classical spacetime and gravity mentioned above. Furthermore, motivated by the 'full covariance' and 'inherent quantumness' of our theory, we draw numerous close parallels between our scenario and certain ideas of Einstein about the so-called (post general relativity) 'new ether' concept, the unitary field theory that goes hand in hand with the latter, but more importantly, about the possible abandonement altogether, in the light of singularities and quantum discontinuities, of this continuous field theory and the $\mathcal{C}^\infty$-spacetime continuum supporting it for "*a purely algebraic description of reality*" [38]. *In toto*, we argue that ADG, especially in its finitary and causal application to Lorentzian quantum gravity in the present paper, may provide the basis for the "*organic*" [36], "*algebraic*" [38] theory that Einstein was searching for in order to replace the multiply assailed by unmanageable singularities, unphysical infinities and other anomaliles geometric spacetime continuum of macroscopic physics. At the same time, we will maintain that this abandonement of the spacetime manifold for a more finitistic-algebraic theory can be captured to a great extent by the mathematical notion of Gel'fand duality—a notion that permeates the general sheaf-theoretic methods of ADG effectively ever since its inception [65, 66, 67] as well its particular finitary, causal and quantal applications thereafter [92, 86, 87, 74, 93, 75, 88, 89, 90]. The paper concludes with some remarks on $\mathcal{C}^\infty$-smooth singularities—some of which having already been presented in a slightly different, purely ADG-theoretic, guise in [73]—that anticipate a paper currently in preparation [76].

2 Connections in Abstract Differential Geometry

Connections, *alias* 'generalized differentials', are *the* central objects in ADG which purports to abstract from, thus axiomatize and effectively generalize, the usual differential calculus on $\mathcal{C}^\infty$-manifolds. In this section we give a brief *résumé* of both the local and global ADG-theoretic perspective on linear (Koszul), pseudo-Riemannian (Lorentzian) connections and their associated curvatures. For more details and completeness of exposition, the reader is referred to [67, 68, 71].

2.1 Basic Definitions about Linear Connections

The main notion here is that of *differential triad* $\mathfrak{T} = (\mathbf{A}_X, \partial, \mathbf{\Omega}(X))$, which consists of a sheaf $\mathbf{A}_X$ of (complex) abelian algebras A over an in general *arbitrary topological space* X called the

structure sheaf or the *sheaf of coefficients* of the triad,[10] a sheaf $\mathbf{\Omega}$ of (differential) A-modules Ω over X, and a $\mathbf{C}$-derivation ∂ defined as the *sheaf morphism*

$$\partial : \mathbf{A} \longrightarrow \mathbf{\Omega} \tag{1}$$

which is $\mathbf{C}$-linear and satisfies Leibniz's rule

$$\partial(s \cdot t) = s \cdot \partial(t) + t \cdot \partial(s) \tag{2}$$

for any local sections s and t of $\mathbf{A}$ (*ie*, $s, t \in \Gamma(U, \mathbf{A}) \equiv \mathbf{A}(U)$, with $U \subseteq X$ open). It can be shown that the usual differential operator ∂ in (1) above is *the* prototype of a *flat* $\mathbf{A}$-*connection* [67, 68].

The aforementioned generalization of the usual differential operator ∂ to an (abstract) $\mathbf{A}$-connection $\mathcal{D}$ involves two steps emulating the definition of ∂ above. First, one identifies $\mathcal{D}$ with a suitable ($\mathbf{C}$-linear) sheaf morphism as in (1), and second, one secures that the Leibniz condition is satisfied by $\mathcal{D}$, as in (2) above. So, given a differential triad $\mathfrak{T} = (\mathbf{A}, \partial, \mathbf{\Omega})$, let $\mathcal{E}$ be an $\mathbf{A}$-module sheaf on X. Then, the first step corresponds to defining $\mathcal{D}$ as a map

$$\mathcal{D} : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathbf{A}} \mathbf{\Omega} \cong \mathbf{\Omega} \otimes_{\mathbf{A}} \mathcal{E} \equiv \mathbf{\Omega}(\mathcal{E}) \tag{3}$$

which is a $\mathbf{C}$-linear morphism of the complex vector sheaves involved, while the second, that this map satisfies the following condition

$$\mathcal{D}(\alpha \cdot s) = \alpha \cdot \mathcal{D}(s) + s \otimes \partial(\alpha) \tag{4}$$

for $\alpha \in \mathbf{A}(U)$, $s \in \mathcal{E}(U) \equiv \Gamma(U, \mathcal{E})$, and U open in X.

The connection $\mathcal{D}$ as defined above may be coined a '*Koszul linear connection*' and its existence on the vector sheaf $\mathcal{E}$ is crucially dependent on both the base space X and the structure sheaf $\mathbf{A}$. For X a *paracompact* and *Hausdorff* topological space, and for $\mathbf{A}_X$ a *fine* sheaf on it, the existence of $\mathcal{D}$ is well secured, as for instance in the case of $\mathcal{C}^\infty$-smooth manifolds [67, 68].

2.1.1 The local form of $\mathcal{D}$

Given a local gauge $e^U \equiv \{U;\ (e_i)_{0 \leq i \leq n-1}\}$ of the vector sheaf $\mathcal{E}$ of rank n,[11] every continuous local section $s \in \mathcal{E}(U)$ ($U \in \mathcal{U}$) can be expressed as a unique superposition $\sum_{i=1}^n s_i e_i$ with coefficients

[10] The pair $(X, \mathbf{A}_X)$ is called a $\mathbf{C}$-*algebraized space*, where $\mathbf{C}$ corresponds to the constant sheaf of the complex numbers $\mathbb{C}$ over X, which is naturally injected into $\mathbf{A}_X$ (*ie*, $\mathbf{C} \overset{\subseteq}{\to} \mathbf{A}_X$ and, plainly, $\mathbb{C} = \Gamma(X, \mathbf{C}) \equiv \mathbf{C}(X)$). It is tacitly assumed that for every open set U in X, the algebra $\mathbf{A}(U)$ of continuous local sections of $\mathbf{A}_X$ is a unital, commutative and associative algebra over $\mathbb{C}$. It must be noted here however that one could start with a $\mathbf{K}$-algebraized space ($\mathbf{K} = \mathbf{R}, \mathbf{C}$) in which the structure sheaf $\mathbf{A}_X$ would consist of unital, abelian and associative algebras over the fields $\mathbb{K} = \mathbb{R}, \mathbb{C}$ respectively. Here we have just fixed $\mathbb{K}$ to the complete field of complex numbers, but in the future we are going to discuss also the real case. Also, in either case $\mathbf{A}_X$ is assumed to be *fine*. In the sequel, when it is rather clear what the base topological space X is, we will omit it from $\mathbf{A}_X$ and simply write $\mathbf{A}$.

[11] We recall from [67, 68, 75] that in ADG, $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ is called *a local frame* or *a coordinatizing open cover of*, or even *a local choice of basis (or gauge!) for* $\mathcal{E}$. The e_is in e^U are local sections of $\mathcal{E}$ (*ie*, elements of $\Gamma(U, \mathcal{E})$) constituting a basis of $\mathcal{E}(U)$. We also mention that for the $\mathbf{A}$-module sheaf $\mathcal{E}$, regarded as a vector sheaf of rank n, one has by definition the following $\mathbf{A}|_U$-isomorphisms: $\mathcal{E}|_U = \mathbf{A}^n|_U = (\mathbf{A}|_U)^n$ and, concomitantly, the following

s_i in $\mathbf{A}(U)$. The action of $\mathcal{D}$ on these sections reads

$$\mathcal{D}(s) = \sum_{i=1}^{n}(s_i\mathcal{D}(e_i) + e_i \otimes \partial(s_i)) \tag{5}$$

with

$$\mathcal{D}(e_i) = \sum_{i=1}^{n} e_i \otimes \omega_{ij},\ 1 \leq i, j \leq n \tag{6}$$

for some unique $\omega_{ij} \in \mathbf{\Omega}(U)$ $(1 \leq i, j \leq n)$, which means that $\omega \equiv (\omega_{ij}) \in M_n(\mathbf{\Omega}(U)) = M_n(\mathbf{\Omega})(U)$ is an $n \times n$ matrix of sections of local 1-forms. Thus, (5) reads via (6)

$$\mathcal{D}(s) = \sum_{i=1}^{n} e_i \otimes (\overbrace{\partial(s_i)}^{\partial} + \sum_{i=1}^{n} \overbrace{s_j\omega_{ij}}^{\omega}) \equiv (\partial + \omega)(s) \tag{7}$$

So that, *in toto*, every connection $\mathcal{D}$ can be written locally as

$$\mathcal{D} = \partial + \omega \tag{8}$$

with (8) effectively expressing the procedure commonly known in physics as *localizing* or *gauging* the usual (flat) differential ∂ to the (curved) *covariant derivative* $\mathcal{D}$. Thus, the (non-flat) ω part of $\mathcal{D}$, called *the gauge potential* in physics, measures the deviation from differentiating flatly (*ie*, by ∂), when one differentiates 'covariantly' by $\mathcal{D}$.[12]

2.1.2 Local gauge transformations of $\mathcal{D}$

We investigate here, in the context of ADG, the behavior of the gauge potential part $\mathcal{A}$ of $\mathcal{D}$ under local gauge transformations—the so-called '*transformation law of potentials*' in [67, 68].

Thus, let $\mathcal{E}$ be an $\mathbf{A}$-module or a vector sheaf of rank n. Let $e^U \equiv \{U;\ e_{i=1\cdots n}\}$ and $f^V \equiv \{V;\ f_{i=1\cdots n}\}$ be local gauges of $\mathcal{E}$ over the open sets U and V of X which, in turn, we assume have non-empty intersection $U \cap V$. Let us denote by $g \equiv (g_{ij})$ the following *change of local gauge matrix*

$$f_j = \sum_{i=1}^{n} g_{ij}e_i \tag{9}$$

equalities section-wise: $\mathcal{E}(U) = \mathbf{A}^n(U) = \mathbf{A}(U)^n$ (with $\mathbf{A}^n$ the n-fold Whitney sum of $\mathbf{A}$ with itself). Thus, $\mathcal{E}$ is *a locally free* $\mathbf{A}$*-module of finite rank* n—an appellation synonymous to *vector sheaf* in ADG [67, 68]. For $n = 1$, the vector sheaf $\mathcal{E}$ is called a *line sheaf* and it is symbolized by $\mathcal{L}$.

[12]In the sequel we will symbolize the gauge potential part of $\mathcal{D}$ in (8) by $\mathcal{A}$ instead of ω in order to be in agreement with our notation in the previous papers [74, 75], as well as with the standard notation for the spin-Lorentzian connection in current Lorentzian quantum gravity research [6, 7, 8, 14].

which, plainly, is a local (*ie*, relative to $U \cap V$) section of the 'natural' structure group sheaf $\mathcal{GL}(n, \mathbf{A})$ of $\mathcal{E}$[13]—that is, $g_{ij} \in \mathrm{GL}(n, \mathbf{A}(U \cap V)) = \mathcal{GL}(n, \mathbf{A})(U \cap V)$.

Without going into the details of the derivation, which can be found in [67, 68], we note that under such a local gauge transformation g, the gauge potential part $\omega \equiv \mathcal{A}$ of $\mathcal{D}$ in (8) transforms as follows

$$\mathcal{A}^{'} = g^{-1}\mathcal{A}g + g^{-1}\partial g \tag{10}$$

a way we are familiar with from the usual differential geometry of the smooth fiber bundles of gauge theories. For completeness, it must be noted here that, in (10), $\mathcal{A} \equiv (\mathcal{A}_{ij}) \in M_n(\mathbf{\Omega}^1(U)) = M_n(\mathbf{\Omega}^1)(U)$ and $\mathcal{A}^{'} \equiv (\mathcal{A}^{'}_{ij}) \in M_n(\mathbf{\Omega}^1(V)) = M_n(\mathbf{\Omega}^1)(V)$. The transformation of $\mathcal{A}$ under local gauge changes is called *affine* or *inhomogeneous* in the usual gauge-theoretic parlance precisely because of the term $g^{-1}\partial g$. We will return to this affine term in subsection 2.3 and subsequently in section 5 where we will comment on the essentially non-geometrical (*ie*, non-tensorial) character of connection. Also, anticipating our discussion of moduli spaces of gauge-equivalent connections in the next section, we note that (10) expresses an equivalence relation '$\stackrel{g}{\sim}$' between the gauge potentials $\mathcal{A}$ and $\mathcal{A}^{'}$.

2.2 Pseudo-Riemannian (Lorentzian) Metric Connections

In this subsection we are interested in endowing a vector sheaf $\mathcal{E}$ of finite rank $n \in \mathbb{N}$ with an indefinite $\mathbf{A}$-valued symmetric inner product ρ, and, concomitantly, study $\mathbf{A}$-connections $\mathcal{D}$ that are compatible with the (indefinite) metric g associated with ρ—the so-called *metric connections*. With an eye towards the applications to Lorentzian (quantum) gravity in the sequel, we are particularly interested in metric $\mathcal{D}$s relative to Lorentzian metrics of signature $\mathrm{diag}(g) = (-, +, +, \cdots)$. Also, continuing our work [74] which dealt with *principal Lorentzian finsheaves of qausets*, we are interested in the *group sheaves* $\mathcal{A}ut_{\mathbf{A}}(\mathcal{E})$ *of* $\mathbf{A}$*-automorphisms of* $\mathcal{E}$—the *principal sheaves of structure symmetries of* $\mathcal{E}$.[14] In the case of a real (*ie*, $\mathbb{K} = \mathbb{R}$ and $\mathbf{R}$-algebraized space) Lorentzian vector sheaf $(\mathcal{E}, \rho)$ of rank 4,[15] the stalks of the corresponding $\mathcal{G}$-sheaves will 'naturally'

[13] We will present some rudiments of structure group (or principal or $\mathcal{G}$-) sheaves of associated vector sheaves $\mathcal{E}$ in the next subsection. One may recognize $\mathcal{GL}(n, \mathbf{A})$ above as the local version of the automorphism group sheaf $\mathcal{A}ut\mathcal{E}$ of $\mathcal{E}$. The adjective 'local' here pertains to the fact mentioned earlier that ADG assumes that $\mathcal{E}$ is locally isomorphic to $\mathbf{A}^n$.

[14] Commonly known as $\mathcal{G}$-sheaves in the mathematical literature [67].

[15] We would like to declare up-front that in this paper we provide no argument whatsoever for assuming that the dimensionality (rank) n of our vector sheaves is the 'empirical' (or better, 'conventional') 4 of the spacetime manifold of 'macroscopic experience' (or better, of the classical theory). In the course of this work the reader will realize that all our constructions are manifestly independent of the classical 4-dimensional, locally Euclidean, $\mathcal{C}^\infty$-smooth, Lorentzian spacetime manifold of general relativity so that we will time and again doubt whether the latter, and the host of (mathematical) structures that classically it is thought of as carrying (*eg*, its uncountably infinite cardinality of events, its dimensionality, its topological, differential and metric structures), is a physically meaningful concept. For example, we will maintain that dimensionality and the metric are free mathematical choices of (*ie*, fixed by) the theorist and not Nature's own, while that the topology and differential structure are inherent in the dynamical objects (fields) that may be thought of as living and propagating on 'spacetime', not by that inert background 'spacetime' itself, which is devoid of any physical meaning. Moreover, all this will be

be assumed to host the group $SO(1,3)^{\uparrow}$—the orthochronous Lorentz group of (local) isometries of $(\mathcal{E},\rho)$ which, in turn, is locally isomorphic to the spin-group $SL(2,\mathbb{C})$.[16] We thus catch a first glimpse of the spin-Lorentzian connections considered in the context of curved finsheaves of qausets in [74], which will be dealt with in more detail in section 4.

Thus, let $\mathcal{E}$ be a vector sheaf. By an **A**-*valued pseudo-Riemannian inner product* ρ *on* $\mathcal{E}$ (over X) we mean a *sheaf* morphism

$$\rho:\ \mathcal{E}\oplus\mathcal{E}\longrightarrow\mathbf{A} \tag{11}$$

which is i) **A**-*bilinear* between the **A**-modules concerned, ii) symmetric (*ie*, $\rho(s,t)=\rho(t,s),\ s,t\in\mathcal{E}(U)$) and of indefinite signature, as well as iii) *strongly non-degenerate*. That is, we assume that $\rho(s,t)$, for any two local sections s and t in $\mathcal{E}(U)$,[17] is given via the canonical isomorphism

$$\mathcal{E}\overset{\tilde{\rho}}{\cong}\mathcal{E}^{*} \tag{12}$$

between $\mathcal{E}$ and its dual $\mathcal{E}^{*}$, as

$$\tilde{\rho}(s)(t):=\rho(s,t) \tag{13}$$

with (12) being true up to an **A**-isomorphism[18].

We further assume that for the vector sheaf $\mathcal{E}$ (of finite rank $n\in\mathbb{N}$) endowed with the **A**-connection $\mathcal{D}$, the vector sheaf $\mathbf{\Omega}$ in the given differential triad $\mathfrak{T}=(\mathbf{A},\partial,\mathbf{\Omega})$ is the dual of $\mathcal{E}$ appearing in (12) (*ie*, $\mathbf{\Omega}=\mathcal{E}^{*}\equiv\mathcal{H}om_{\mathbf{A}}(\mathcal{E},\mathbf{A})$). Thus, in line with the usual Christoffel theory [67, 68], we can define a *linear connection* ∇, as follows

$$\nabla:\ \mathcal{E}\times\mathcal{E}\longrightarrow\mathcal{E} \tag{14}$$

---

expressed in an algebraic, locally finite setting quite remote from the uncountable continuous infinity of events of the manifold.

[16]In the sense that their corresponding Lie algebras are isomorphic: $so(1,3)^{\uparrow}\simeq sl(2,\mathbb{C})$ [74].

[17]It is important to notice here that the **A**-metric ρ is not a (bilinear) map assigned to the points of the base space X *per se* (which is only assumed to be a topological, not a differential, let alone a metric, space), but to the fibers (stalks) of the relevant module or vector sheaves which are inhabited by the geometrical objects that live on X. As noted in a previous footnote, in our scheme, metric and, as we shall see later, topological and differential properties concern the objects that live on 'space(time)', not the supporting space(time) itself. This recalls Gauss' and Riemann's original labors with endowing the linear fiber spaces tangent to a sphere with a bilinear quadratic form—a metric. They ascribed a metric to the linear fibers, not to the supporting sphere itself which, anyway, is manifestly 'non-linear' [73]. What we wish to highlight by these remarks is that *space(time) carries no metric*. Equally important is to note that the **A**-valued metric ρ is imposed on these objects *by us* and it is intimately tied to (*ie*, takes values in) our own measurements (arithmetics) in **A** (see comparison between the notions of connection and curvature in 2.3.5). ρ is not a property of space(time), which does not exist (in a physical sense) anyway; rather, it is an attribute related to our own measurements of 'it all'. These remarks are important for our subsequent physical interpretation of ADG in its application to finitary Lorentzian quantum gravity in the next four sections. It is a preliminary indication that in our theory the base space(time) is an ether-like 'substance' without any physical significance. See remarks about 'gravity as a gauge theory' in the next section, about the 'physical insignificance' or 'non-physicality' of spacetime in 5.1.1 and about 'the relativity of differentiability' in 6.2, as well as some similar anticipations in [74, 75].

[18]The epithet 'strongly' to 'non-degenerate' above indicates that $\tilde{\rho}$ in (12) is also *onto*.

acting section-wise on $\mathcal{E}(U)$ as

$$\nabla(s,t) \equiv \nabla_s(t) := \mathcal{D}(t)(s) \tag{15}$$

Now, one says that $\mathcal{D}$ *is a pseudo-Riemannian* $\mathbf{A}$*-connection* or that it is compatible with the indefinite metric g of the inner product ρ in (11), whenever it fulfills the following two conditions:

- *Riemannian symmetry*: $\nabla(s,t) - \nabla(t,s) = [s,t]$; for $s,t \in \mathcal{E}(U)$ and $[\,.\,,\,.\,]$ the usual Lie bracket (product).

- *Ricci identity*: $\partial(\rho(s,t))(u) = \rho(\nabla(u,s),t) + \rho(s,\nabla(u,t))$; for $s,t,u \in \mathcal{E}(U)$, as usual.

In particular, for a Lorentzian ρ and its associated g,[19] an $\mathbf{A}$-connection $\mathcal{D}$ is said to be compatible with the Lorentz $\mathbf{A}$-inner product ρ on $\mathcal{E}$[20] when its associated Christoffel ∇ in (14) satisfies

$$\nabla\rho = 0 \tag{16}$$

which, in turn, is equivalent to the following '*horizontality*' condition for the canonical isomorphism $\tilde{\rho}$ in (12) relative to the *connection* $\mathcal{D}_{\mathcal{E}\otimes_{\mathbf{A}}\mathcal{E}^*}$ in the tensor product vector sheaf $\mathcal{H}om_{\mathbf{A}}(\mathcal{E},\mathcal{E}^*) = (\mathcal{E}\otimes_{\mathbf{A}}\mathcal{E})^* = \mathcal{E}^*\otimes_{\mathbf{A}}\mathcal{E}^*$ induced by the $\mathbf{A}$-connection $\mathcal{D}$ on $\mathcal{E}$

$$\mathcal{D}_{\mathcal{H}om_{\mathbf{A}}(\mathcal{E},\mathcal{E}^*)}(\tilde{\rho}) = 0 \tag{17}$$

It is worth reminding the reader who is familiar with the usual theory that (17) above implies that the Levi-Civita $\mathbf{A}$-connection $\mathcal{D}$ induced by the Lorentz $\mathbf{A}$-metric ρ is *torsion-free* [70].

2.2.1 Connections on (Lorentzian) principal sheaves

As mentioned in the beginning of this subsection, of special interest in our study is the case of a (real) Lorentzian vector sheaf $(\mathcal{E},\rho)$ of rank 4 whose $\mathbf{A}$-automorphism sheaf $\mathcal{A}ut_{\mathbf{A}}\mathcal{E}^{\uparrow}$ bears $G = L^{\uparrow} := SO(1,3)^{\uparrow}$—the orthochronous ρ-preserving $\mathbf{A}$-automorphisms of $\mathcal{E}$ in its stalks.[21] $\mathfrak{L}^+$ is the *principal sheaf of structure symmetries of* $\mathcal{E}^{\uparrow}$. In turn, $\mathcal{E}^{\uparrow}$ is called the $\mathfrak{L}^+$*-associated vector sheaf*.[22]

But let us first give a brief discussion of connections on principal sheaves *à la* ADG and then focus on spin-Lorentzian (metric) connections. The reader will have to wait until section 4 where

[19] With respect to a *local (coordinate) gauge* $e^U \equiv \{U;\ (e_i)_{0\leq i\leq n-1}\}$ of the vector sheaf $\mathcal{E}$ of rank n, $\rho(e_i,e_j) = g_{ij} = \mathrm{diag}(-1,+1,\cdots)$ [67, 68].

[20] Such a metric connection is commonly known as *Levi-Civita connection*.

[21] One may wish to symbolize the pair $(\mathcal{E},\rho)$ by $\mathcal{E}^{\uparrow}$, thus $\mathcal{A}ut_{\mathbf{A}}\mathcal{E}^{\uparrow}$ by $\mathfrak{L}^+$. In the sequel, when it is clear from the context that we are talking about a Lorentzian vector sheaf $\mathcal{E}^{\uparrow} = (\mathcal{E},\rho)$, we may use the symbols $\mathcal{E}$ and $\mathcal{E}^{\uparrow}$ for it interchangeably hopefully without confusion. For a general vector sheaf $\mathcal{E}$, $\mathcal{A}ut_{\mathbf{A}}\mathcal{E}$ is a subsheaf of $\mathcal{E}nd\mathcal{E}$, in fact, for a given open $U \subseteq X$, $\mathcal{A}ut_{\mathbf{A}}(\mathcal{E})(U) \simeq \mathrm{End}_{\mathbf{A}}(\mathcal{E}|_U)^{\bullet}$—the upper dot denoting *invertible* endomorphisms. We thus write in general: $\mathcal{A}ut_{\mathbf{A}}(\mathcal{E}) \equiv \mathcal{A}ut\mathcal{E} := (\mathcal{E}nd\mathcal{E})^{\bullet}$.

[22] Henceforth we will assume that every principal sheaf acts on the typical stalk of its associated sheaf on the left (see below).

we recall in more detail from [74] the curved principal finsheaves $\vec{\mathcal{P}}_i^{\uparrow}$ of qausets and their non-trivial connections $\vec{\mathcal{D}}_i$. For the material that is presented below, we draw information mainly from [124, 125, 126].

Let $\mathcal{G}$ be a sheaf of groups[23] over X. Let $\mathcal{E}$ be an $\mathbf{A}$-module and σ a representation of $\mathcal{G}$ in $\mathcal{E}$, that is to say, a *a continuous group sheaf morphism*

$$\sigma : \mathcal{G} \longrightarrow \mathcal{A}ut\mathcal{E} \tag{18}$$

effecting local (*ie*, U-wise in X) continuous left-actions of $\mathcal{G}$ on $\mathcal{E}$ as follows

$$\mathcal{G}(U) \times \mathcal{E}(U) \longrightarrow \mathcal{E} \;:\; (g, v) \rightarrowtail [\sigma(g)](v),\; v \in \mathcal{E}(U),\; g \in \mathcal{G}(U) \tag{19}$$

Also, by letting $\mathbf{\Omega}^1$ be a sheaf of (first order) differential $\mathbf{A}$-modules over $\mathcal{E}$, $\mathbf{\Omega}^1(\mathcal{E}) := \mathbf{\Omega}^1 \otimes_{\mathbf{A}} \mathcal{E}$ as in (3), we define a *Lie sheaf of groups* $\mathcal{G}$[24] to be the quadruple $(\mathfrak{L}, \mathcal{E}, \sigma, \dot{\partial})$, where $\mathfrak{L}$ is an $\mathbf{A}$-module of Lie algebras,[25] σ a representation of $\mathfrak{L}$ in $\mathcal{E}$, and $\dot{\partial}$ the following $\mathbf{A}$-module sheaf morphism

$$\dot{\partial} : \mathfrak{L} \longrightarrow \mathbf{\Omega}^1(\mathcal{E}) \tag{20}$$

which reminds one of the flat connection ∂ in (1). $\dot{\partial}$, called the *Maurer-Cartan differential of* $\mathcal{G}$ *relative to* σ,[26] satisfies

$$\dot{\partial} : (s \cdot t) = \sigma(t^{-1}) \cdot \dot{\partial} s + \dot{\partial} t \tag{21}$$

It must be noted here that in the same way that ADG—the differential geometry of vector sheaves—represents an abstraction and generalization of the usual calculus on vector bundles over $\mathcal{C}^\infty$-smooth manifolds to the effect that *no calculus, in the usual sense, is employed at all* [67, 68], Lie sheaves of groups are the abstract analogues of the usual Lie groups that play a central role in the classical differential geometry of principal fiber bundles over differential manifolds [124, 125, 126].

Thus, let $\mathcal{G}$ be a Lie sheaf of groups as above. Formally speaking, by a *principal sheaf* $\mathcal{P}$ *with structure group* $\mathcal{G}$ *relative to* $\mathcal{G} = (\mathfrak{L}, \mathcal{E}, \sigma, \dot{\partial})$[27] we mean a quadruple $(\mathcal{P}, \mathfrak{L}, X, \pi)$ consisting of a sheaf of sets $\mathcal{P}$[28] such that:

[23]By abuse of notation, and hopefully without confusing the reader, in the sequel we will also symbolize the groups that dwell in the stalks of $\mathcal{G}$ by '$\mathcal{G}$'.

[24]The reader should note that in the present paper we symbolize the gauge (structure) group of both Y-M theory and gravity also by $\mathcal{G}$, hopefully without causing any confusion between it and the abstract Lie sheaf of groups above.

[25]By assuming that the group sheaf $\mathcal{G}$ in (18) is a sheaf of Lie groups, we may take $\mathfrak{L}$ to be the corresponding sheaf of Lie algebras.

[26]$\dot{\partial}$ is also known as the *logarithmic differential* of $\mathcal{G}$.

[27]Where $\mathfrak{L}$ is the sheaf of Lie algebras of the Lie group sheaf $\mathcal{G}$. $\mathfrak{L}$ is supposed to represent the *local structural type* of $\mathcal{P}$ [125].

[28]$\mathcal{P}$ may be thought of as 'coordinatizing' the principal sheaf, thus we use the same symbol '$\mathcal{P}$' for the principal sheaf and its coordinatizing sheaf of sets. π is the usual projection map from $\mathcal{P}$ to the base space X. For more details, refer to [124, 125, 126].

1. There is a continuous right-action of $\mathcal{L}$ on $\mathcal{P}$.
2. There is an open gauge $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of X and isomorphisms of sheaves of sets (*ie*, coordinate mappings)

$$\phi_\alpha : \; \mathcal{P}|_{U_\alpha} \xrightarrow{\cong} \mathcal{L}|_{U_\alpha} \tag{22}$$

satisfying

$$\phi_\alpha(s \cdot g) = \phi_\alpha(s) \cdot g; \;\; s \in \mathcal{P}(U_\alpha), g \in \mathcal{L}(U_\alpha) \tag{23}$$

Given $\mathcal{P}$, a vector sheaf $\mathcal{E}$ and the representation $\sigma : \; \mathcal{L} \longrightarrow \mathcal{A}ut\mathcal{E}$, one obtains the so-called *associated sheaf of* $\sigma(\mathcal{P})$,[29] which is a sheaf of vector spaces locally of type $\mathcal{E}$ in the sense that, relative to a coordinate gauge $\mathcal{U}$ for X, there are coordinate maps

$$\Phi_\alpha : \; \sigma(\mathcal{P})|_{U_\alpha} \xrightarrow{\cong} \mathcal{E}|_{U_\alpha} \tag{24}$$

We assume that the associated vector sheaves $\mathcal{E}$ of the $\mathcal{G}$-sheaves $\mathcal{P}$ presented above are of the type mentioned before in the context of ADG, namely, *locally free* $\mathbf{A}$*-modules of finite rank* (*ie*, locally isomorphic to $\mathbf{A}^n$) [67, 68]. We thus come to the main definition of a connection $\dot{\mathcal{D}}$ on a principal sheaf $\mathcal{P}$ generalizing the Maurer-Cartan differential $\dot{\partial}$ in (20) in a way analogous to how $\mathcal{D}$ on a vector sheaf $\mathcal{E}$ in (3) generalized the flat differential ∂ in (1). Thus,

$$\dot{\mathcal{D}} : \; \mathcal{P} \longrightarrow \mathbf{\Omega}^1(\mathcal{E})^{30} \tag{25}$$

is a morphism of sheaves of sets satisfying

$$\dot{\mathcal{D}}(s \cdot g) = \sigma(g^{-1}) \cdot \dot{\mathcal{D}}s + \dot{\partial}g; \;\; s \in \mathcal{P}(U) \text{ and } g \in \mathcal{L}(U) \tag{26}$$

Locally (*ie*, U-wise in X), one can show, in complete analogy to the local decomposition $\partial + \mathcal{A}$ of the $\mathbf{A}$-connection $\mathcal{D}$ on $\mathcal{E}$ in(8), that $\dot{\mathcal{D}}$ too can be written as

$$\dot{\mathcal{D}} = \dot{\partial} + \dot{\mathcal{A}} \tag{27}$$

and that, for a given coordinate gauge $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ for X with *natural local coordinate sections of* $\mathcal{P}$ $s_\alpha := \phi_\alpha^{-1} \circ \mathbf{1}|_{U_\alpha} \in \mathcal{P}(U_\alpha)$,

$$(\dot{\mathcal{A}})_\alpha = \dot{\mathcal{D}}(s_\alpha) \in \mathbf{\Omega}^1(\mathcal{E})(U_\alpha) \tag{28}$$

in complete analogy to the local gauge potential 1-forms $\mathcal{A}$ of connections $\mathcal{D}$ on vector sheaves presented in (5)–(8).[31]

[29] Otherwise called *the* $\mathcal{P}$-, or even, *the* $\mathcal{L}$*-associated vector sheaf.*

[30] This morphism can be equivalently written as $\dot{\mathcal{D}} : \; \mathcal{P} \longrightarrow \mathbf{\Omega}^1 \otimes_{\mathbf{A}} \mathcal{L}(\equiv \mathbf{\Omega}^1(\mathcal{L}))$, to manifest the usual statement that a connection on a principal sheaf is a Lie algebra-valued 1-form. Time and again we will encounter this definition below.

[31] Furthermore, one can show that for a local change of gauge g as in (9), the $\dot{\mathcal{A}}$s obey a transformation law of potentials completely analogous to the one obeyed by the $\mathcal{A}$s in (10). Without going into any details, it reads: $\dot{\mathcal{A}}' = \sigma(g)^{-1}\dot{\mathcal{A}}\sigma(g) + \sigma(g)^{-1}\dot{\partial}g, \; (\sigma(g^{-1}) \equiv \sigma(g)^{-1})$ [124, 125, 126].

Now, the essential point in this presentation of connections $\dot{\mathcal{D}}$ on principal sheaves $\mathcal{P}$ in relation to our presentation of $\mathbf{A}$-connections $\mathcal{D}$ on vector sheaves $\mathcal{E}$ earlier, is that when the latter are the $\mathcal{P}$-associated sheaves relative to corresponding representations $\sigma:\ \mathcal{L}\longrightarrow \mathcal{A}ut\mathcal{E}$, the following 'commutative diagram' may be used to picture formally the *'σ-induced projection $\hat{\sigma}$'* of $\dot{\mathcal{D}}$ on $\mathcal{P}$ to $\mathcal{D}$ on $\mathcal{E}$

$$\begin{tikzcd} \mathcal{P} \arrow[r, "\hat{\sigma}"] \arrow[d, "\dot{\mathcal{D}}"'] & \mathbf{A} \arrow[d, "\mathcal{D}"] \\ \mathbf{\Omega}^1(\mathcal{L}) \arrow[r, "\mathrm{id}"'] & \mathbf{\Omega}^1(\mathcal{E}) \end{tikzcd} \tag{29}$$

where $\hat{\sigma}$ may be regarded a morphism between $\mathcal{P}$ and $\mathbf{A}$ regarded simply as sheaves of structureless sets.[32]

To make an initial contact with [74], we can now particularize the general ADG-based presentation of principal sheaves $\mathcal{P}$ above to (real) *Lorentzian $\mathcal{G}$-sheaves*. As briefly noted earlier, the structure group G dwelling in the stalks of the latter is taken to be $L^{\uparrow} := SO(1,3)^{\uparrow}$—the Lie group of orthochronous Lorentz $\mathbf{A}$-isometries, so that $\mathcal{P}$ in this case is denoted by $\mathfrak{L}^{+}$. The $\mathfrak{L}^{+}$-associated sheaf $\mathcal{E}^{\uparrow}=(\mathcal{E},\rho)$ is a (real) vector sheaf of rank 4, equipped with an $\mathbf{A}$-metric ρ of absolute trace equal to 2. Thus, there is a local homomorphism (representation) σ of the Lie algebra $so(1,3)^{\uparrow}\simeq sl(2,\mathbb{C})$ of the structure group $L^{\uparrow}$ in $\mathfrak{L}^{+}$ into the 'Lie algebra' sheaf $aut_{\mathbf{A}}(\mathcal{E}^{\uparrow})$ of the group sheaf $\mathcal{A}ut_{\mathbf{A}}(\mathcal{E}^{\uparrow})$ of invertible $\mathbf{A}$-endomorphisms of $\mathcal{E}$ preserving the Lorentzian $\mathbf{A}$-metric ρ—that is, the $\mathbf{A}$-metric ρ symmetries (isometries) of $\mathcal{E}^{\uparrow}$.

Collecting information from our presentation of connections on $\mathcal{G}$-sheaves and their associated vector sheaves, we are in a position now to recall from [74] that, in the particular case of the $\mathfrak{L}^{+}$-associated vector sheaf $\mathcal{E}^{\uparrow}$,

> *the gauge potential part $\mathcal{A}$ of an $\mathbf{A}$-connection $\mathcal{D}$ on $\mathcal{E}^{\uparrow}$ is an $so(1,3)^{\uparrow}\simeq sl(2,\mathbb{C})$-valued 1-form on $\mathfrak{L}^{+}$.*

the so-called *spin-Lorentzian connection 1-form*.

After we discuss the affine space A of Y-M and Lorentzian gravitational $\mathcal{G}$-connections from an ADG-theoretic perspective in 2.4, as well as present the connection-based vacuum Einstein equations ADG-theoretically in the next section, we are going to return to the kinematical spin-Lorentzian connections on principal finsheaves of qausets and their associated vector sheaves

[32] That is to say, by forgetting both the group structure of the $\mathcal{G}$-sheaf $\mathcal{P}$ and the algebra structure of the structure sheaf $\mathbf{A}$. The inverse procedure of building the principal sheaf $\mathcal{P}$ and the connection $\dot{\mathcal{D}}$ on it from its associated vector sheaf $\mathcal{E}$ and the connection $\mathcal{D}$ on it, may be loosely called *'σ-induced lifting $\hat{\sigma}^{-1}$'* of $(\mathcal{E},\mathcal{D})$ to $(\mathcal{P},\dot{\mathcal{D}})$. The σ^{-1}-lifting is a forgetful correspondence since, in going from a vector sheaf to its structure group sheaf, the linear structure of the former is lost—something which is in fact reflected on that, while $\mathcal{D}$ is $\mathbf{C}$-linear, $\dot{\mathcal{D}}$ is not. However, for more details about commutative diagrams like (29) between principal sheaves $(\mathcal{P}_1,\dot{\mathcal{D}}_1)$ and $(\mathcal{P}_2,\dot{\mathcal{D}}_2)$, their corresponding associated sheaves $(\mathcal{E}_1,\mathcal{D}_1)$ and $(\mathcal{E}_1,\mathcal{D}_1)$, as well as the respective projections $\hat{\sigma}$ of the former to the latter, the reader is referred to [126].

studied in [74] in section 4, then we will formulate their dynamical vacuum Einstein equations in 5, and finally, in the same section, we will discuss a possible covariant (*ie*, action-based, path integral-type of) quantum dynamics for them.

2.3 Curvatures of A-Connections

In ADG, the curvature R of an $\mathbf{A}$-connection $\mathcal{D}$, like $\mathcal{D}$ itself, is defined as an $\mathbf{A}$-*module sheaf morphism*. More analytically, let $\mathfrak{T} = (\mathbf{A}, \partial, \mathbf{\Omega})$ be a differential triad as before. Define 'inductively' the following hierarchy of sheaves of $\mathbb{Z}_+$-graded $\mathbf{A}$-modules $\mathbf{\Omega}^i$ ($i \in \mathbb{Z}_+ \equiv \mathbb{N} \cup \{0\}$) of exterior (*ie*, Cartan differential) forms over X

$$\mathbf{\Omega}^0 := \mathbf{A},\ \mathbf{\Omega} \equiv \mathbf{\Omega}^1 := \mathbf{A} \wedge_{\mathbf{A}} \mathbf{\Omega},\ \mathbf{\Omega}^2 = \mathbf{A} \wedge_{\mathbf{A}} \mathbf{\Omega}^1 \wedge_{\mathbf{A}} \mathbf{\Omega}^1, \cdots \mathbf{\Omega}^i \equiv (\mathbf{\Omega}^1)^i := \wedge^i_{\mathbf{A}} \mathbf{\Omega}^1 \tag{30}$$

and, in the same way that $\partial(\equiv d^0)$ is a $\mathbf{C}$-linear morphism between $\mathbf{A} \equiv \mathbf{\Omega}^0$ and $\mathbf{\Omega} \equiv \mathbf{\Omega}^1$ as depicted in (1), define a second differential operator $\mathtt{d}(\equiv d^1)$ again as the following $\mathbf{C}$-linear $\mathbf{A}$-module sheaf morphism

$$\mathtt{d} :\ \mathbf{\Omega}^1 \longrightarrow \mathbf{\Omega}^2 \tag{31}$$

obeying relative to ∂

$$\mathtt{d} \circ \partial = 0 \text{ and } \mathtt{d}(\alpha \cdot s) = \alpha \cdot \mathtt{d}s - s \wedge \partial\alpha,\ (\alpha \in \mathbf{A}(U), s \in \mathbf{\Omega}(U), U \text{ open in } X) \tag{32}$$

and called *the 1st exterior derivation.*[33]

Then, in complete analogy to the 'extension' of the flat connection ∂ to $\mathtt{d}$ above, given a $\mathbf{A}$-module $\mathcal{E}$ endowed with an $\mathbf{A}$-connection $\mathcal{D}$, one can define the *1st prolongation of* $\mathcal{D}$ to be the following $\mathbf{C}$-linear vector sheaf morphism

$$\mathcal{D}^1 :\ \mathbf{\Omega}^1(\mathcal{E}) \longrightarrow \mathbf{\Omega}^2(\mathcal{E}) \tag{33}$$

satisfying section-wise relative to $\mathcal{D}$

$$\mathcal{D}^1(s \otimes t) := s \otimes \mathtt{d}t - t \wedge \mathcal{D}s,\ (s \in \mathcal{E}(U), t \in \mathbf{\Omega}^1(U), U \text{ open in } X) \tag{34}$$

We are now in a position to define the curvature R of an $\mathbf{A}$-connection $\mathcal{D}$ by the following commutative diagram

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "\mathcal{D}"] \arrow[dr, "R \equiv \mathcal{D}^1 \circ \mathcal{D}"'] & & \mathbf{\Omega}^1(\mathcal{E}) \equiv \mathcal{E} \otimes_{\mathbf{A}} \mathbf{\Omega}^1 \arrow[dl, "\mathcal{D}^1"] \\ & \mathbf{\Omega}^2(\mathcal{E}) \equiv \mathcal{E} \otimes_{\mathbf{A}} \mathbf{\Omega}^2 & \end{tikzcd} \tag{35}$$

[33]In (30), '$\wedge_{\mathbf{A}}$' is the completely antisymmetric $\mathbf{A}$-respecting tensor product '$\otimes_{\mathbf{A}}$'.

from which we read that

$$R \equiv R(\mathcal{D}) := \mathcal{D}^1 \circ \mathcal{D} \tag{36}$$

Therefore, any time we have the $\mathbf{C}$-linear morphism $\mathcal{D}$ and its prolongation $\mathcal{D}^1$ at our disposal, we can define the curvature $R(\mathcal{D})$ of the connection $\mathcal{D}$.[34] By defining a *curvature space* as the finite sequence $(\mathbf{A}, \partial, \mathbf{\Omega}^1, \mathtt{d}, \mathbf{\Omega}^2)$ of $\mathbf{A}$-modules and $\mathbf{C}$-linear morphisms between them, we can distill the last statement to the following:

> *we can always define the curvature R of a given $\mathbf{A}$-connection $\mathcal{D}$, provided we have a curvature space.*

As a matter of fact, it is rather straightforward to see that, for $\mathcal{E}$ a vector sheaf, $R(\mathcal{D})$ is an $\mathbf{A}$-morphism of $\mathbf{A}$-modules, in the following sense

$$\begin{gathered} R \in \mathrm{Hom}_{\mathbf{A}}(\mathcal{E}, \mathbf{\Omega}^2(\mathcal{E})) = \mathcal{H}om_{\mathbf{A}}(\mathcal{E}, \mathbf{\Omega}^2(\mathcal{E}))(X) \\ \mathbf{\Omega}^2(\mathcal{E}nd\mathcal{E})(X) = Z^0(\mathcal{U}, \mathbf{\Omega}^2(\mathcal{E}nd\mathcal{E})) \end{gathered} \tag{37}$$

where $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ is an open cover of X and $Z^0(\mathcal{U}, \mathbf{\Omega}^2(\mathcal{E}nd\mathcal{E}))$ the $\mathbf{A}(U)$-module of 0-*cocycles* of $\mathbf{\Omega}^2(\mathcal{E}nd\mathcal{E})$ relative to the $\mathcal{U}$-coordinatization of X.[35]

2.3.1 The local form of R

Motivated by (37) and the last remarks, we are in a position to give the local form for the curvature R of a given $\mathbf{A}$-connection $\mathcal{D}$. Thus, let $\mathcal{E}$ be a vector sheaf of rank n, $\mathcal{D}$ an $\mathbf{A}$-connection on it and $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ a local coordinatization frame of it. By virtue of (37) we have

$$\begin{gathered} R(\mathcal{D}) = R = (R_{ij}^{(\alpha)}) \equiv ((R_{ij}^{\alpha})) \in Z^0(\mathcal{U}, \mathbf{\Omega}^2(\mathcal{E}nd\mathcal{E})) \\ \subseteq \prod_\alpha \mathbf{\Omega}^2(\mathcal{E}nd\mathcal{E})(U_\alpha) = \prod_\alpha M_n(\mathbf{\Omega}^2(U_\alpha)) \end{gathered} \tag{38}$$

so that we are led to remark that:

> *the curvature R of an $\mathbf{A}$-connection $\mathcal{D}$ on a vector sheaf $\mathcal{E}$ of rank n is a 0-cocycle of local $n \times n$ matrices having for entries local sections of $\mathbf{\Omega}^2$—ie, local 2-forms on X.*

[34] In connection with (36), one can justify our earlier remark that the standard differential operator ∂, regarded as an $\mathbf{A}$-connection as in (1) (*ie*, as the sheaf morphism $\partial : \mathbf{A} \longrightarrow \mathbf{\Omega}^1 = \mathbf{A} \otimes_{\mathbf{A}} \mathbf{\Omega}^1 \equiv \mathbf{\Omega}^1(\mathbf{A})$), is *flat*, since: $R(\partial) = \mathtt{d}\circ\partial = d^1 \circ d^0 \equiv d^2 = 0$ (which is secured by the nilpotency of the usual Cartan-Kähler (exterior) differential operator d [75]). In the latter paper, and in a sheaf-cohomological fashion, it was shown that it is exactly $\mathcal{D}$'s deviation from nilpotency (*ie*, from flatness), which in turn *defines* a non-vanishing curvature $R(\mathcal{D}) = \mathcal{D}^2 \neq 0$, that prevents a sequence $\cdots \xrightarrow{\mathcal{D}^{i-1}} \mathbf{\Omega}^i \xrightarrow{\mathcal{D}^i} \mathbf{\Omega}^{i+1} \xrightarrow{\mathcal{D}^{i+1}} \cdots$ of differential $\mathbf{A}$-module sheaves $\mathbf{\Omega}^i$ and $\mathbf{C}$-linear sheaf morphisms $\mathcal{D}^i$ between them from being a *complex*. ($\mathcal{D}^i$, $i \geq 2$, stand for high-order prolongations of the $\mathcal{D}^0 \equiv \mathcal{D}$ and $\mathcal{D}^1$ connections above [67, 68].)

[35] One may wish to recall that, for a vector sheaf $\mathcal{E}$ like the one involved in (37), $\mathcal{E}nd\mathcal{E} \equiv \mathcal{H}om_{\mathbf{A}}(\mathcal{E}, \mathcal{E}) \cong \mathcal{E} \otimes_{\mathbf{A}} \mathcal{E}^* = \mathcal{E}^* \otimes_{\mathbf{A}} \mathcal{E}$.

2.3.2 Local gauge transformations of R

We wish to investigate here the behavior of the curvature $R(\mathcal{D})$ of an $\mathbf{A}$-connection $\mathcal{D}$ under local gauge transformations—the so-called '*transformation law of field strengths*' in the usual gauge-theoretic parlance and in ADG [67, 68].

Thus, let $g \equiv g_{ij} \in \mathcal{GL}(n, \mathbf{A})(U \cap V)$ be the change-of-gauge matrix we considered in (9) in connection with the transformation law of gauge potentials. Again, without going into the details of the derivation, we bring forth from [67, 68] the following local transformation law of gauge field strengths

$$\begin{aligned}\text{for a local frame change: } e^U \xrightarrow{g} e^V (U, V \text{ open gauges in } X),\\ \text{the curvature transforms as: } R \xrightarrow{g} R' = g^{-1} R g\end{aligned} \tag{39}$$

which we are familiar with from the usual differential geometric (*ie*, smooth fiber bundle-theoretic) treatment of gauge theories. For completeness, we remind ourselves here that, in (39), $R^{U\cap V} \equiv (R_{ij}^{U\cap V}) \in M_n(\mathbf{\Omega}^2(U \cap V))$—an $n \times n$ matrix of sections of local 2-forms. The transformation of R under local gauge changes is called *homogeneous* or *covariant* in the usual gauge-theoretic parlance. We will return to this term in 2.3.5 and subsequently in section 5 where we will comment on the geometrical (*ie*, tensorial) character of curvature.

2.3.3 Cartan's structural equation—Bianchi identities

We wish to express in ADG-theoretic terms certain well known, but important, (local) identities about curvature. We borrow material mainly from [68].

So, let $\mathcal{E}$ be a vector sheaf and assume that $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ provides a coordinatization for it, as above. The usual Cartan's structural equation reads in our case

$$R^{(\alpha)} \equiv (R_{ij}^{(\alpha)}) = \mathtt{d}\mathcal{A}^{(\alpha)} + \mathcal{A}^{(\alpha)} \wedge \mathcal{A}^{(\alpha)} \in M_n(\mathbf{\Omega}^2(U_\alpha)) \tag{40}$$

and similarly in the case of a sheaf $\mathcal{E}$ of $\mathbf{A}$-modules and U open in X

$$R = \mathtt{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A};\ (\mathcal{A}_{ij}) \in M_n(\mathbf{\Omega}^1(U)) \tag{41}$$

(41) can be also written in the *Maurer-Cartan form*

$$R = \mathtt{d}\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}] \tag{42}$$

by setting $[\mathcal{A}, \mathcal{A}] \equiv \mathcal{A} \wedge \mathcal{A} - \mathcal{A} \wedge \mathcal{A}$. For a one-dimensional vector sheaf $\mathcal{E}$ (*ie*, a line sheaf $\mathcal{L}$) equipped with an $\mathbf{A}$-connection $\mathcal{D}$, the commutator in (41) vanishes and we obtain the curvature as the following 0-cocycle

$$R = (\mathtt{d}\mathcal{A}_a) \in Z^0(\mathcal{U}, \mathtt{d}\mathbf{\Omega}^1) = (\mathtt{d}\mathbf{\Omega}^1)(X) \subseteq \mathbf{\Omega}^2(X) \subseteq \prod_\alpha \mathbf{\Omega}^2(U_\alpha) \tag{43}$$

with $(\mathcal{A}_\alpha) \in C^0(\mathcal{U}, \mathbf{\Omega}^1) = \prod_\alpha \Omega^1(U_\alpha)$ the corresponding (local) $\mathbf{A}$-connection 0-cochain of $\mathcal{D}$.

To express the familiar Bianchi identities obeyed by the curvature $R(\mathcal{D})$, and similarly to the extension of $\partial \equiv d^0$ to the nilpotent Cartan-Kähler differential $\mathtt{d} \equiv d^1$ in 2.3, we need the extension of d^1 to a *second exterior derivation* $\mathbf{d} \equiv d^2$ which again is a $\mathbf{C}$-linear sheaf morphism of the respective exterior $\mathbf{A}$-modules[36]

$$\mathbf{d}: \ \mathbf{\Omega}^2 \longrightarrow \mathbf{\Omega}^3 \tag{44}$$

acting (local) section-wise as follows

$$\mathbf{d}(s \wedge t) := \mathtt{d}s \wedge t - s \wedge \mathtt{d}t, \ \forall s, t \in \mathbf{\Omega}^1(U); \ U \subseteq X \text{ open} \tag{45}$$

and being nilpotent

$$d^2 \circ d^1 \equiv d \circ d \equiv d^2 = 0 \tag{46}$$

As a result of the extension of $\mathtt{d}$ to $\mathbf{d}$, the aforementioned *curvature space* $(\mathbf{A}, \partial, \mathbf{\Omega}^1, \mathtt{d}, \mathbf{\Omega}^2)$, when enriched with the $\mathbf{A}$-module sheaf $\mathbf{\Omega}^3$ as well as with the nilpotent $\mathbf{C}$-linear morphism $\mathbf{d}$ in (44), becomes a so-called *Bianchi space*.

In a Bianchi space, the usual *second Bianchi identity* holds

$$\mathbf{d}R \equiv dR = [R, \mathcal{A}] \equiv R \wedge \mathcal{A} - \mathcal{A} \wedge R \tag{47}$$

where $\mathbf{d}$ is understood to effect coordinate-wise: $\mathbf{d}: \ M_n(\mathbf{\Omega}^2) \longrightarrow M_n(\mathbf{\Omega}^3)$.

In the case of a line sheaf $\mathcal{L}$, one can easily show by using (30) and the nilpotency of d that

$$dR = 0 \tag{48}$$

which is usually referred to as the *homogeneous field equation*. The latter, in turn, translates to the following cohomological statement:

> *the curvature R of an $\mathbf{A}$-connection $\mathcal{D}$ on a line sheaf $\mathcal{L}$ over X provides a closed 2-form on X.*

which came very handy in the sheaf-cohomological classification of the curved associated line sheaves of qausets and their quanta—the so-called '*causons*'—performed in [75].

Finally, one can also show that the second prolongation $\mathcal{D}^2_{\mathcal{E}nd\mathcal{E}}$ of the induced $\mathbf{A}$-connection $\mathcal{D}_{\mathcal{E}nd\mathcal{E}}$ on $\mathcal{E}nd\mathcal{E} \cong \mathcal{E} \otimes_{\mathbf{A}} \mathcal{E}^*$ satisfies the following 'covariant version' of the second Bianchi identity (47) above

$$\mathcal{D}^2_{\mathcal{E}nd\mathcal{E}}(R) = 0 \tag{49}$$

where $\mathcal{D}^2_{\mathcal{E}nd\mathcal{E}}: \ \mathbf{\Omega}^2(\mathcal{E}nd\mathcal{E}) \longrightarrow \mathbf{\Omega}^3(\mathcal{E}nd\mathcal{E})$. Thus, similarly to (47), one also shows that

$$\mathcal{D}_{\mathcal{E}nd\mathcal{E}} R = dR + [\mathcal{A}, R] \tag{50}$$

[36]In the sequel, following the cohomological custom in [75], we identify ∂, $\mathtt{d}$ and $\mathbf{d}$ (and all higher order exterior derivations) with the generic Cartan differential d, specifying its order only when necessary and by writing generically d^i ($i \geq 0$).

which proves the *equivalence* of the second (exterior differential) Bianchi identity on $\mathcal{E}$ and its induced (covariant differential) version on $\mathcal{E}nd\mathcal{E}$.

2.3.4 The Ricci tensor, scalar and the Einstein-Lorentz (curvature) space

Given a (real) Lorentzian vector sheaf $(\mathcal{E}, \rho)$ of rank n equipped with a non-flat $\mathbf{A}$-connection $\mathcal{D}$,[37], one can define, in view of (37) the following *Ricci curvature operator* $\mathcal{R}$ relative to a local gauge U of $\mathcal{E}$

$$\mathcal{R}(\,.\,, s)t \in (\mathcal{E}nd\mathcal{E})(U) = M_n(\mathbf{A}(U)) \tag{51}$$

for local sections s and t of $\mathcal{E}$ in $\mathcal{E}(U) = \mathbf{A}^n(U) = \mathbf{A}(U)^n$. $\mathcal{R}$ is an $\mathcal{E}nd\mathcal{E}$-valued operator.[38]

Since $\mathcal{R}$ is matrix-valued, as (51) depicts, one can take its trace thus define the following *Ricci scalar curvature operator* $\mathcal{R}$

$$\mathcal{R}(s,t) := tr(\mathcal{R}(\,.\,, s)t) \tag{52}$$

which, plainly, is $\mathbf{A}(U)$-valued.

We have built a suitable conceptual background to arrive now at a central notion in this paper. A (real) Lorentzian vector sheaf $\mathcal{E}^{\uparrow} = (\mathcal{E}, \rho)$ over a $\mathbf{R}$-algebraized space $(X, \mathbf{A})$ such that:

1. it is supported by a differential triad $\mathfrak{T} = (\mathbf{A}, \partial, \mathbf{\Omega}^1)$ relative to which (12) holds, that is, $\mathcal{E}^* \equiv \mathbf{\Omega}^1$,

2. there is a $\mathbf{R}$-linear Lorentzian connection $\mathcal{D}$ on it satisfying (17) (*ie*, a metric connection) and, furthermore,

3. it is a curvature space $(\mathbf{A}, \partial, \mathbf{\Omega}^1, \mathtt{d}, \mathbf{\Omega}^2)$ supporting a *null* $\mathcal{R}$, that is to say, *a Ricci scalar operator satisfying the vacuum Einstein equations*

$$\mathcal{R}(\mathcal{E}) = 0 \tag{53}$$

is called an *Einstein-Lorentz (E-L) space*, while the corresponding base space X, an *Einstein space* [70].[39] Of course, it has been implicitly assumed that, for an appropriate choice of structure sheaf $\mathbf{A}$, equation (53) *can be actually derived from the variation of the corresponding Lagrangian density* (*alias*, Einstein-Hilbert action functional $\mathfrak{EH}$). We will return to this assumption in the next section.

In connection with the definition of an Einstein space X, it is worth noting that

[37] The reader should note that below, and only in the vacuum Einstein case, we will symbolize the connections involved by $\mathcal{D}$ instead of the calligraphic $\mathcal{D}$ we have used so far to denote the general $\mathbf{A}$-connections in ADG.

[38] Due to this, $\mathcal{R}$ has been called a *curvature endomorphism* in [70].

[39] In the next section, where we will cast Lorentzian gravity as a Y-M-type of gauge theory *à la* ADG, we will also define a *Yang-Mills space* analogous to the Einstein space above.

> *the only structural requirement that ADG places on the Einstein base space X is that it is, merely, a topological space—in fact, an arbitrary topological space, without any assumptions whatsoever about its differential, let alone its metric, structure.*

This prompts us to emphasize, once again [67, 68, 70, 77, 78, 74, 75, 73], the essential 'working philosophy' of ADG:

> *to actually do differential geometry one need not assume any 'background differentiable space' X, for differentiability derives from the algebraic structure of the objects (structure algebras) that live on that 'space'. The only role of the latter is a secondary, auxiliary and, arguably, a 'physically atrophic' one in comparison to the active role played by those objects (in particular, the algebra $\mathbf{A}(U)$ of local sections of $\mathbf{A}$) themselves: X merely provides an inert, ether-like scaffolding for the localization and the dynamical interactions ('algebraically and sheaf-theoretically modelled interrelations') of those physically significant objects—a passive substrate of no physical significance whatsoever, since it does not actively participate into the algebraico-dynamical relations between the objects themselves.*[40] *All in all, the basic objects that ADG works with is the sections of the sheaves in focus—that is, the entities that live in the stalks of the relevant sheaves, and not with the underlying base space X, so that any notion of 'differentiability' according to ADG derives its sense from the algebraic relations between (ie, the algebraic structure of) those (local) sections, with the apparently 'intervening between' or 'permeating through these objects' background space X playing absolutely no role in it.*

2.3.5 A fundamental difference between $\mathcal{D}$ and $R(\mathcal{D})$ and its physical interpretation

At this point it is worth stressing a characteristic difference between an $\mathbf{A}$-connection $\mathcal{D}$ and its curvature $R(\mathcal{D})$—a difference that is emphasized by ADG, it has a significant bearing on the physical interpretation of our theory, and it has been already highlighted in both [74] and [75]; namely that,

while R is an $\mathbf{A}$-morphism, $\mathcal{D}$ is only a $\mathbf{K}$-morphism $(\mathbf{K} = \mathbf{R}, \mathbf{C})$.

This means that, since the structure sheaf $\mathbf{A}$ corresponds to 'geometry' in our algebraic scheme, in the sense that $\mathbf{A}(U)$—the algebra of local sections of $\mathbf{A}$—represents *the algebra of local operations*

[40] Its arbitrary character—again, X is assumed to be simply an *arbitrary topological space*—reflects precisely its physical insignificance. This non-physicality, the 'algebraic inactivity' and 'dynamically non-participatory character' so to speak, of the background space will become transparent subsequently when we formulate the dynamical equations for vacuum gravity entirely in terms of *sheaf morphisms between the objects—ie, virtually the sections—that live on X* (the main sheaf morphism being the connection $\mathcal{D}$—arguably the central operator with which one actually does differential geometry!). At this point we would like to further note, according to [67], that *a sheaf morphism is* actually reduced to *a family of (local) morphisms between* (the complete presheaves of) *local sections* $Mor(\mathcal{E},\mathcal{F}) \ni \phi \longleftrightarrow (\phi_U) \in Mor(\Gamma(\mathcal{E}),\Gamma(\mathcal{F})$—a *category equivalence* through (the *section functor*) Γ. In the last section we will return to the inert, passive, ether-like character of the base space in the particular case that X is (a region of) a $\mathcal{C}^\infty$-smooth spacetime manifold. There we will argue how *ADG 'relativizes' the 'differential properties' of space(time).*

of measurement (of the quantum system 'space-time') *relative to the local laboratory* (frame, or gauge, or even 'observation device') U [87, 74, 75], it effectively encodes *our* geometrical information about the physical system in focus.[41] Consequently,

> *R, which, being an $\mathbf{A}$-morphism, respects our local measurements—the 'geometry-encoding (measuring) apparatus' $\mathbf{A}$ of ADG so to speak—is a geometrical object (ie, a tensor) in our theory and lies on the classical side of the quantum divide. On the other hand, $\mathcal{D}$, which respects only the constant sheaf $\mathbf{K}(=\mathbf{R},\mathbf{C})$ but not our (local) measurements in $\mathbf{A}$, is not a geometrical object[42] and it lies on the quantum (ie, the purely algebraic, à la Leibniz* [73]*), side of Heisenberg's cut.*[43]

2.4 The Affine Space A of A-Connections

We fix the $\mathbf{K}$-algebraized space $(X, \mathbf{A})$ and the differential triad $\mathfrak{T} = (\mathbf{A}, \partial, \mathbf{\Omega})$ on it with which we are working, and we let $\mathcal{E}$ be an $\mathbf{A}$-module on X. We denote by

$$\mathsf{A}_{\mathbf{A}}(\mathcal{E}) \tag{54}$$

the set of $\mathbf{A}$*-connections on* $\mathcal{E}$. By definition (3), $\mathsf{A}_{\mathbf{A}}(\mathcal{E})$ is a subset of $\mathrm{Hom}_{\mathbf{K}}(\mathcal{E}, \mathbf{\Omega}(\mathcal{E}))$ $(\mathbf{\Omega} \equiv \mathbf{\Omega}^1)$ whose zero element may be regarded as the zero $\mathbf{A}$-connection in $\mathsf{A}_{\mathbf{A}}(\mathcal{E})$. However, by (4), one infers that ∂ is also zero in this case, thus we will exclude altogether the zero $\mathbf{A}$-connection from

[41] As mentioned before, $\mathbf{A}_X$ is the abelian algebra sheaf of '*generalized arithmetics*' in ADG generalizing the usual commutative coordinate sheaf ${}^{\mathbb{R}}\mathcal{C}^{\infty}_M$ of the smooth manifold—the sheaf of abelian rings ${}^{\mathbb{R}}\mathcal{C}^{\infty}(M)$ of infinitely differentiable, real-valued functions on the differential manifold M. We tacitly assume in our theory that *'geometry' is synonymous to 'measurement'*; hence, in the quantum context, it is intimately related to 'observation' (being, in fact, the result of it). Furthermore, since the results of observation arguably lie on the classical side of the quantum divide (the so-called *Heisenberg Schnitt*), $\mathbf{A}$ must be a sheaf of *abelian* algebras. This is supposed to be a concise ADG-theoretic encodement of Bohr's correspondence principle, namely, that *the numbers that we obtain upon measuring the properties of a quantum mechanical system (the so-called q-numbers) must be commutative (the so-called c-numbers).* In other words, *the acts of measurement yield c-numbers from q-numbers*, so that 'geometry'—the structural analysis of (the algebras of our local measurements of) 'space'—deals, by definition, with commutative numbers and the (sheaves of) abelian algebras into which the latter are effectively encoded. See also closing remarks in [68] for a similar discussion of 'geometry *à la* ADG' in the sense above, as well as our remarks about *Gel'fand duality* in 5.5.1

[42] Another way to say this is that *the notion of connection is algebraic (ie, analytic), not geometrical.* In short, *$\mathcal{D}$ is not a tensor.* That R is a tensor while $\mathcal{D}$ is not is reflected in their (local) gauge transformation laws that we saw earlier: $\mathcal{A}$ transforms affinely or inhomogeneously (non-tensorially), while R covariantly or homogeneously (tensorially) under a (local) change of gauges.

[43] Although it must be also stressed that $\mathcal{D}$, like the usual notion of derivative ∂ that it generalizes, has a *geometrical interpretation.* As the derivative of a function (of a single variable) is usually interpreted in a Newtonian fashion as the slope (gradient) of the tangent to the curve (graph) of the function, so $\mathcal{D}$ can be interpreted geometrically as a parallel transporter of objects (here, $\mathbf{A}$-tensors) along geometrical curves (paths) in space(time). However, it is rather inappropriate to think of $\mathcal{D}$ as a geometrical object proper and at the same maintain a geometrical interpretation for it, for *does it not sound redundant to ask for the geometrical interpretation of an 'inherently geometrical' object, like the triangle or the circle, for instance?* In other words, *if the notion of connection was 'inherently geometrical', it would certainly be superfluous to also have a geometrical interpretation for it.*

$\mathsf{A}_{\mathbf{A}}(\mathcal{E})$. Since any connection may be taken to serve as an 'origin' for the space of $\mathbf{A}$-connections, we conclude that

> $\mathsf{A}_{\mathbf{A}}(\mathcal{E})$ *is an affine space modelled after the* $\mathbf{A}(X)$*-module* $\mathrm{Hom}_{\mathbf{K}}(\mathcal{E}, \mathbf{\Omega}(\mathcal{E}))$. *For a vector sheaf* $\mathcal{E}$*,* $\mathrm{Hom}_{\mathbf{K}}(\mathcal{E}, \mathbf{\Omega}(\mathcal{E}))$ *becomes* $\mathbf{\Omega}(\mathcal{E}nd\mathcal{E})(X)$.

Now, in connection with the statement above, let $\mathcal{D}$ be an $\mathbf{A}$-connection in $\mathsf{A}_{\mathbf{A}}(\mathcal{E}) \equiv \mathrm{Hom}_{\mathbf{K}}(\mathcal{E}, \mathbf{\Omega}(\mathcal{E}))$. Then, it can be shown [67, 68] that any other connection $\mathcal{D}'$ in $\mathsf{A}_{\mathbf{A}}(\mathcal{E})$ is of the form

$$\mathcal{D}' = \mathcal{D} + u \tag{55}$$

for a uniquely defined $u \in \mathrm{Hom}_{\mathbf{A}}(\mathcal{E}, \mathbf{\Omega}^1(\mathcal{E}))$. For $\mathcal{E}$ a vector sheaf, u belongs to $\mathbf{\Omega}^1(\mathcal{E}nd\mathcal{E})(X)$. Thus, for a given $\mathcal{D} \in \mathsf{A}_{\mathbf{A}}(\mathcal{E})$ we can formally write (55) as: $\mathsf{A}_{\mathbf{A}}(\mathcal{E}) = \mathcal{D} + \mathrm{Hom}_{\mathbf{A}}(\mathcal{E}, \mathbf{\Omega}^1(\mathcal{E}))$, within a bijection. Interestingly enough, (55) tells us that the difference of two connections, which are $\mathbf{K}$-linear sheaf morphisms, is an $\mathbf{A}$-morphism like the curvature; hence, in view of the comparison between $\mathcal{D}$ and $R(\mathcal{D})$ above, we can say that $\mathcal{D}' - \mathcal{D}$ is a geometrical object since it respects our measurements in $\mathbf{A}$ by transforming homogeneously (tensorially) under (local) gauge transformations.[44]

In the particular case of a line sheaf $\mathcal{L}$,

> $\mathsf{A}_{\mathbf{A}}(\mathcal{L})$ *can be identified with* $\mathbf{\Omega}^1(X)$*—the* $\mathbf{A}(X)$*-module of '*1*-forms' on* X.

Thus, given any connection $\mathcal{D}$ in $\mathsf{A}_{\mathbf{A}}(\mathcal{L})$, any other connection $\mathcal{D}'$ on $\mathcal{L}$ can be written as $\mathcal{D}' = \mathcal{D} + \omega$ for some unique ω in $\mathbf{\Omega}^1(X)$. This result was used in [75] for the sheaf-cohomological classification of the line sheaves associated with the curved principal finsheaves of qausets and the non-trivial connections on them in [74].

We will return to $\mathsf{A}_{\mathbf{A}}(\mathcal{E})$ in the next section where we will factor it by the structure (gauge) group $\mathcal{G} = \mathrm{Aut}(\mathcal{E})$ of $\mathcal{E}$ to obtain the orbifold or moduli space $\mathsf{A}_{\mathbf{A}}(\mathcal{E})/\mathcal{G}$ of gauge-equivalent connections on $\mathcal{E}$ of a Y-M or gravitational type depending on $\mathcal{G}$.

3 Vacuum Einstein Gravity as a Y-M-type of Gauge Theory à la ADG

In this section we present the usual vacuum Einstein gravity in the language of ADG, that is, as a Y-M-type of gauge theory describing the dynamics of a Lorentzian connection on a suitable principal Lorentzian sheaf and its associated vector sheaf, in short, on an E-L space as defined above. We present only the material that we feel is relevant to our subsequent presentation of finitary vacuum Lorentzian gravity encouraging the reader to refer to [67, 68, 70, 71] for more analytical treatment of Y-M theories and gravity *à la* ADG. But let us first motivate in a rather general way this conception of *gravity as a gauge theory*.

[44] The reader could verify that u transforms covariantly under (local) changes of gauge.

3.1 Physical Motivation

It is well known that the original formulation of general relativity was in terms of a pseudo-Riemannian metric $g_{\mu\nu}$ on a $\mathcal{C}^\infty$-smooth spacetime manifold M. For Einstein, the ten components of the metric represented the gravitational potentials—the pure gravitational dynamical degrees of freedom so to speak. However, very early on it was realized that there was an equivalent formulation of general relativity involving the dynamics of the so-called *spin-connection* ω. This approach came to be known as *Einstein-Cartan theory* [46] and arguably it was the first indication, long before the advent of the Y-M gauge theories of matter, that gravity concealed some sort of gauge invariance which was simply masked by the metric formulation.[45] In fact, Feynman, in an attempt to view gravity purely field-theoretically and, *in extenso*, quantum gravity as a quantum field theory (*ie*, in an attempt to quantize gravity using a language and techniques more familiar to a particle physicist than a general relativist,[46]) he essentially 'downplayed', or at least undermined, the differential geometric picture of general relativity and instead he concentrated on its gauge-theoretic attributes. Brian Hatfield nicely reconstructed Feynman's attitude towards (quantum) gravity in [40],[47] as follows

> "...Thus it is no surprise that Feynman would recreate general relativity from a non-geometrical viewpoint. The practical side of this approach is that one does not have to learn some 'fancy-schmanzy' (as he liked to call it) differential geometry in order to study gravitational physics. (Instead, one would just have to learn some quantum field theory.) However, when the ultimate goal is to quantize gravity, Feynman felt that the geometrical interpretation just stood in the way. From the field theoretic viewpoint, one could avoid actually defining—up front—the physical meaning of quantum geometry, fluctuating topology, space-time foam, *etc.*, and instead look for the geometrical meaning after quantization...Feynman certainly felt that the geometrical interpretation is marvellous, but '*the fact that a massless spin-2 field can be interpreted as a metric was simply a coincidence that might be understood as representing some kind of gauge invariance'*."[48]

Feynman's 'negative' attitude towards the standard differential geometry and the smooth spacetime continuum that supports it,[49] especially if we consider the unrenormalizable infinities that plague quantum gravity when treated as another quantum field theory, is quite understandable if we recall from the beginning of the present paper his earlier position—repeated once again,

[45] Recently, after reading [60], the present authors have become aware of a very early attempt by Eddington at formulating general relativity (also entertaining the possibility of unifying gravity with electromagnetism) based solely on the affine connection and not on the metric, which is treated as a secondary structure, 'derivative' in some sense from the connection. Indicatively, Kostro writes: "*...*[Eddington's] *approach relied on affine geometry. In this geometry, connection, and not metric, is considered to be the basic mathematical entity. The metric $g_{\mu\nu}(x)$ needed for the description of gravitational interactions, appears here as something secondary, which is derived from connection...*" (bottom of page 99 and references therein).

[46] Such an approach was championed a decade later by Weinberg in a celebrated book [127].

[47] See Hatfield's preamble titled '*Quantum Gravity*'.

[48] Our emphasis of Feynman's words as quoted by Hatfield.

[49] The reader must have realized by now that by the epithets 'standard', or 'usual', or more importantly, 'classical', to 'differential geometry' we mean the differential geometry of $\mathcal{C}^\infty$-smooth manifolds—the so-called 'calculus on differential manifolds'.

that "*the theory that space is continuous is wrong, because we get…infinities…the simple ideas of geometry, extended down to infinitely small, are wrong!*".[50]

However, it must be noted that Feynman's 'unconventional' attempt in the early 60s to tackle the problem of quantum gravity gauge quantum field-theoretically was preceded by Bergmann's ingenious recasting of the Einstein-Cartan theory in terms of 2-component spinors, thus effectively showing that the main dynamical field involved in that theory—the spin connection ω—is an $sl(2,\mathbb{C})$-valued 1-form [16].[51] All in all, it is remarkable indeed that such a connection-based approach to general relativity, classical or quantum, has been revived in the last fifteen years or so in the context of *non-perturbative canonical quantum gravity.* We refer of course to Ashtekar's modification of the Palatini *vierbein* or comoving 4-frame-based formalism by using new canonical variables to describe the phase space of general relativity and in which variables the gravitational constraints are significantly simplified [3]. Interestingly enough, and in relation to Bergmann's work mentioned briefly above, in Ashtekar's scheme the principal dynamical variable is an $sl(2,\mathbb{C})$-valued *self-dual spin-Lorentzian connection* 1*-form* $\mathcal{A}^{+}$[52] [3].

But after this lengthy preamble let us get on with our main aim in this section to present the classical vacuum Lorentzian gravity as a Y-M-type of gauge theory in the manner of ADG.

3.2 Y-M Theory à la ADG—Y-M Curvature Space

Let $(\mathcal{E},\rho)$ be a (real) Lorentzian vector sheaf of finite rank n associated with a differential triad $\mathfrak{T}=(\mathbf{A},\partial,\mathbf{\Omega}^1)$, which in turn is associated with the $\mathbf{R}$-algebraized space $(X,\mathbf{A})$,[53] and $\mathcal{D}$ a non-trivial Lorentzian $\mathbf{A}$-connection on it (*ie*, $R(\mathcal{D})\neq 0$). In ADG, the pair $(\mathcal{E},\mathcal{D})$ is generically referred to as a *Y-M field*, the triplet $(\mathcal{E},\rho,\mathcal{D})$ as a *Lorentz-Yang-Mills (L-Y-M) field*, and it has been shown [67, 68, 70] that[54]

> *every Lorentzian vector sheaf yields a (non-trivial) L-Y-M field $(\mathcal{E},\rho,\mathcal{D})$ on X the (non-vanishing) field strength of which is $\mathcal{F}(\mathcal{D})$.*

As in the definition of the E-L space earlier, in case the curvature $\mathcal{F}$ of the connection $\mathcal{D}$ of a L-Y-M field satisfies the free Y-M equations, which we write as follows[55]

$$\delta^2_{\mathcal{E}nd\mathcal{E}}(\mathcal{F})=0 \text{ or } \Delta^2_{\mathcal{E}nd\mathcal{E}}(\mathcal{F})=0 \tag{56}$$

[50] In the closing section we will return to comment thoroughly, in the light of ADG, on this remark by Feynman and the similar one of Isham also quoted in the beginning of the paper.

[51] More precisely, in Bergmann's theoretical scenario for classical Lorentzian gravity, $g_{\mu\nu}$ is replaced by a field of four 2×2 Pauli spin-matrices which is locally invariant when conjugated by a member of $SL(2,\mathbb{C})$—the double cover of the Lorentz group.

[52] Later in the present section we will discuss briefly self-dual connections from ADG's point of view.

[53] With X a paracompact Hausdorff topological space and $\mathbf{A}$ a *fine* unital commutative algebra sheaf (over $\mathbb{R}$) on it, as usual.

[54] In the sequel, and similarly to how we used different symbols for the (vacuum) gravitational connection $\mathfrak{D}$ and its Y-M counterpart $\mathcal{D}$, we will use $\mathcal{F}$ for the curvature of the latter instead of R ($\mathcal{R}$ and $\mathfrak{R}$) that we used for the former. In the Y-M context the curvature of a connection is usually referred to as the (gauge) field strength.

[55] In (56), 'δ' is the *coderivative* [46] and Δ the *Laplacian operator*, which we will define in an ADG-theoretic manner shortly. These are two equivalent expressions of the free Y-M equations. Their equivalence, which is a consequence of the covariant differential Bianchi identity (50), has been shown in [67].

and which, in turn, we assume that can be obtained from the variation of a corresponding Y-M action functional $\mathfrak{YM}$,[56] the curvature space $(\mathbf{A}, \partial, \mathbf{\Omega}^1, \mathsf{d}, \mathbf{\Omega}^2)$ associated with the L-Y-M field is called a *L-Y-M curvature space*,[57] while the supporting X, a *L-Y-M space*.[58] In connection with the said derivation of the Y-M equations from $\mathfrak{YM}$, we note that[59]

> *the solutions of the Y-M equations that correspond to a given Y-M field $(\mathcal{E}, \mathcal{D})$ are precisely the critical or stationary points (or extrema) of $\mathfrak{YM}$ that can be associated with $\mathcal{E}$.*

In order to make sense of (56) ADG-theoretically, we need to define the coderivative and the Laplacian of a given L-Y-M field $(\mathcal{E}, \rho, \mathcal{D})$. We do this below.

3.2.1 The adjoint δ and the Laplacian Δ of an A-connection in ADG

Let $\mathfrak{T} = (\mathbf{A}, \partial, \mathbf{\Omega}^1)$ be the differential triad we are working with and ρ a Lorentzian **A**-metric on it, as usual. Let also $\mathcal{E}$ be a Lorentzian vector sheaf of finite rank n and $\mathcal{D}$ a Lorentzian Y-M connection on it. By emulating the classical situation sheaf-theoretically, as it is customary in ADG, one can define the *adjoint derivation* δ of $\mathcal{D}$ relative to ρ as the following **A**-morphism of the vector sheaves involved

$$\delta^1 \equiv \delta : \ \mathbf{\Omega}^1(\mathcal{E}) \longrightarrow \mathcal{E}(\equiv \mathbf{\Omega}^0(\mathcal{E})) \tag{57}$$

satisfying

$$\rho(\mathcal{D}(s), t) = \rho(s, \delta(t)) \tag{58}$$

with the obvious identifications: $\forall s \in \mathcal{E}(U)$, $t \in \mathbf{\Omega}^1(\mathcal{E})(U)$, and U a common open gauge of $\mathcal{E}$ and $\mathbf{\Omega}^1(\mathcal{E})$. δ is uniquely defined through the **A**-metric isomorphism $\mathcal{E} \simeq \mathcal{E}^*$ we saw in (12).

To define the Laplacian Δ associated with $\mathcal{D}$, apart from the connection $\mathcal{D} \equiv \mathcal{D}^0$ and the coderivative δ, we also need $\mathcal{D}^1$ (the first prolongation of $\mathcal{D}$, as in (33)) and $\delta^2 : \ \mathbf{\Omega}^2(\mathcal{E}) \longrightarrow \mathbf{\Omega}^1$ (the second contraction relative to $\mathcal{D}$,[60]) as follows

$$\Delta \equiv \Delta^1 := \delta^2 \circ \mathcal{D}^1 + \mathcal{D}^0 \circ \delta^1 \equiv \delta\mathcal{D} + \mathcal{D}\delta : \ \mathbf{\Omega}^1(\mathcal{E}) \longrightarrow \mathbf{\Omega}^1(\mathcal{E}) \tag{59}$$

[56] We will discuss this derivation in more detail shortly.

[57] A particular kind of Bianchi space defined earlier.

[58] In order for the reader not to be misled by our terminology, it must be noted here that, in contrast to the usual term '(free) Yang-Mills field' by which one understands the field strength of a gauge potential which is a solution to the (free) Y-M equations (56), in ADG, admittedly with a certain abuse of language, a Y-M field is just the pair $(\mathcal{E}, \mathcal{D})$, without necessarily implying that $\mathcal{F}(\mathcal{D})$ satisfies (56). On the other hand, the Y-M space X supporting the Y-M curvature space $(\mathbf{A}, \partial, \mathbf{\Omega}^1, \mathsf{d}, \mathbf{\Omega}^2)$ associated with a Y-M field $(\mathcal{E}, \mathcal{D})$, is supposed to refer directly to solutions $\mathcal{F}(\mathcal{D})$ of (56)—as it were, it represents the 'solution space' of (56). This is in complete analogy to the Einstein-Lorentz space and Einstein space X defined in connection with the vacuum Einstein equations for Lorentzian gravity in (53). We will return to comment further on this conception of a curvature space as a geometrical 'solution space' in section 5 when we express (53) in finitary terms.

[59] In fact, the statement that follows is a *theorem* in ADG [68, 70, 71]. We will return to it in 3.3.

[60] Which can be defined in complete analogy to (58).

Higher order Laplacians Δ^i, generically referred to as Δ, can be similarly defined as $\mathbf{K}$-linear vector sheaf morphisms

$$\Delta^i: \ \mathbf{\Omega}^i(\mathcal{E}) \longrightarrow \mathbf{\Omega}^i(\mathcal{E}), \ i \in \mathbb{N} \tag{60}$$

and they read via the corresponding higher order connections $\mathcal{D}^i$ and coderivatives δ^i

$$\Delta^i := \delta^{i+1} \circ \mathcal{D}^i + \mathcal{D}^{i-1} \circ \delta^i, \ i \in \mathbb{N} \tag{61}$$

with the higher order analogues of (58) being

$$\rho(\mathcal{D}^p(s), t) = \rho(s, \delta^{p+1}(t)), \ p \in \mathbb{Z}_+ \tag{62}$$

where ρ is the $\mathbf{A}$-metric on the vector sheaf $\mathbf{\Omega}^p(\mathcal{E})$ and the 'exterior' analogue of (12) reading

$$\mathbf{\Omega}^p(\mathcal{E}) \ \xrightarrow[\tilde{\rho}]{\subset} \ (\mathbf{\Omega}^p(\mathcal{E}))^* \tag{63}$$

Having defined Δ and δ, the reader can now return to (56) understanding $\delta^2_{\mathcal{E}nd\mathcal{E}}$ and $\Delta^2_{\mathcal{E}nd\mathcal{E}}$ as the maps $\delta^2_{\mathcal{E}nd\mathcal{E}}: \ \mathbf{\Omega}^2(\mathcal{E}nd\mathcal{E}) \longrightarrow \mathbf{\Omega}^2(\mathcal{E}nd\mathcal{E})$ and $\Delta^2_{\mathcal{E}nd\mathcal{E}} = \ \delta^3_{\mathcal{E}nd\mathcal{E}} \circ \mathcal{D}^2_{\mathcal{E}nd\mathcal{E}} + \mathcal{D}^1 \circ \delta^2_{\mathcal{E}nd\mathcal{E}}: \ \mathbf{\Omega}^2(\mathcal{E}nd\mathcal{E}) \longrightarrow \mathbf{\Omega}^2(\mathcal{E}nd\mathcal{E})$, respectively.[61] By abusing notation, we may rewrite the free Y-M equations (56) as

$$\delta(\mathcal{F}) = 0 \ \text{ or } \ \Delta(\mathcal{F}) = 0 \tag{64}$$

hopefully without sacrificing understanding.

Our ADG-theoretic exposition of the Y-M equations so far, together with a quick formal comparison that one may wish to make between the aforedefined (vacuum) E-L and the (free) L-Y-M curvature spaces, reveals our central contention in this section, namely that

> *in ADG, vacuum Einstein Lorentzian gravity is a Yang-Mills-type of gauge theory involving the dynamics of a Lorentzian connection $\mathcal{D}$ on an Einstein space X. In complete analogy to the L-Y-M case above, the corresponding triplet $(\mathcal{E}, \rho, \mathcal{D})$ (whose Ricci scalar curvature $\mathcal{R}$ is) satisfying* (53), *is called a (vacuum) Einstein-Lorentz field. For rank $n = 4$, structure group* $\mathrm{Aut}(\mathcal{E}^{\uparrow}) = L^{\uparrow}$ *and principal sheaf $\mathfrak{L}^{+}$, the associated vacuum Einstein-Lorentz field is written as $(\mathcal{E}^{\uparrow}, \mathcal{D})$ $(\mathcal{E}^{\uparrow} = (\mathcal{E}, \rho))$. Locally in the Einstein space X, $\mathcal{D} = \partial + \mathcal{A}$, with $\mathcal{A}$ an $sl(2, \mathbb{C}) \simeq so(1,3)^{\uparrow}$-valued 1-form representing the vacuum gravitational gauge potential.*

3.3 The Einstein-Hilbert Action Functional $\mathfrak{E}\mathfrak{H}$

Now that we have established with the help of ADG the close structural similarity between vacuum Einstein Lorentzian gravity and free Y-M theory, we will elaborate for a while on our remark earlier that both (53) and (56) or (64) derive from the extremization of an action functional—the E-H $\mathfrak{E}\mathfrak{H}$ in the first case, and the Y-M $\mathfrak{Y}\mathfrak{M}$ in the second. Since only vacuum Einstein gravity

[61] Always remembering that the field strength $\mathcal{F}$ of the L-Y-M connection $\mathcal{D}$ is an $\mathbf{A}$-morphism between the $\mathbf{A}$-modules $\mathcal{E}$ and $\mathbf{\Omega}^2(\mathcal{E})$ (*ie*, a member of $\mathcal{H}om_{\mathbf{A}}(\mathcal{E}, \mathbf{\Omega}^2(\mathcal{E}))(X)$), as (37) depicts.

interests us here, we will discuss only the variation of $\mathfrak{EH}$, leaving the variation of $\mathfrak{YM}$ for the reader to read from [67, 68, 71].

As it has been transparent in the foregoing presentation, from the ADG-theoretic point of view, the main dynamical variable in vacuum Einstein Lorentzian gravity is the spin-Lorentzian $\mathbf{A}$-connection $\mathcal{D}$, or equivalently, its gauge potential part $\mathcal{A}$ on the vector sheaf $\mathcal{E}^{\uparrow} = (\mathcal{E}, \rho)$. Thus, one naturally anticipates that

> *the E-H action $\mathfrak{EH}$ is a functional on the affine space $\mathsf{A}_{\mathbf{A}}(\mathcal{E}^{\uparrow})$ of Lorentzian metric (ie, ρ-compatible) $\mathbf{A}$-connections on $\mathcal{E}^{\uparrow}$.*

Indeed, we define $\mathfrak{EH}$ as the following map

$$\mathfrak{EH}: \ \mathsf{A}_{\mathbf{A}}(\mathcal{E}^{\uparrow}) \longrightarrow \mathbf{A}(X) \tag{65}$$

reading 'point-wise'

$$\mathcal{D} \mapsto \mathfrak{EH}(\mathcal{D}) := \mathcal{R}(\mathcal{D}) =: tr\mathcal{R}(\mathcal{D}) \tag{66}$$

where. plainly, $\mathcal{R}$ is a global section of the structure sheaf of coefficients $\mathbf{A}$ (*ie*, $\mathcal{R} \in \mathbf{A}(X)$).

Our main contention (in fact, a theorem in ADG [67, 68, 70]) in 2.3, as well as in 3.2 in connection with Y-M theory, was that

> *the solutions of the vacuum Einstein field equations* (53) *that correspond to a given E-L field* $(\mathcal{E}^{\uparrow}, \mathcal{D})$ *are obtained from extremizing* $\mathfrak{EH}$*—that is, they are the critical or stationary points of the functional* $\mathfrak{EH}$ *associated with* $\mathcal{E}^{\uparrow}$ *in* (65) *and* (66) *above.*

In what follows we will recall briefly how ADG deals with this statement.

The critical points of $\mathfrak{EH}$ can be obtained by first restricting it on a curve $\gamma(t)$ in connection space (*ie*, $\gamma: \ t \in \mathbb{R} \longrightarrow \gamma(t) \in \mathsf{A}_{\mathbf{A}}(\mathcal{E}^{\uparrow})$) and then by infinitesimally varying it around its 'initial' value $\mathfrak{EH}[\mathcal{D}_0] \equiv \mathfrak{EH}[\gamma(0)]$. Alternatively, and following the rationale in [70], in order to find the stationary points of $\mathfrak{EH}$, one has to find the 'tangent vector' at time $t = 0$ to a path $\gamma(t)$ in the affine space $\mathsf{A}_{\mathbf{A}}(\mathcal{E}^{\uparrow})$ of $\mathbf{A}$-connections of $\mathcal{E}^{\uparrow}$, on which path $\mathfrak{EH}$ is constrained to take values in $\mathbf{A}(X)$ as (65) dictates. All in all, one must evaluate

$$\overset{\cdot}{\overbrace{\mathfrak{EH}(\gamma(t))}}\,(0) \equiv \overset{\cdot}{\overbrace{\mathfrak{EH}(\gamma)}}\,(0) \tag{67}$$

where $\dot{x}$ is Newton's notation for $\frac{dx}{dt}$.

For a given Lorentzian metric connection $\mathcal{D}$, one can take the path γ in connection space to be

$$\gamma(t) \equiv \mathcal{D}_t = \mathcal{D} + t\mathfrak{D} \in \mathsf{A}_{\mathbf{A}}(\mathcal{E}^{\uparrow}),\, t \in \mathbb{R} \tag{68}$$

where $\mathfrak{D} \in \mathbf{\Omega}^1(\mathcal{E}nd\mathcal{E}^{\uparrow})(X)$ as mentioned earlier in (55). $\mathcal{D}_t$ may be regarded as the $\mathbf{A}$-connection on $\mathcal{E}^{\uparrow}$ compatible with the Lorentzian metric $\rho_t = \rho + t\rho^{'}$, with $\rho^{'}$ an arbitrary symmetric $\mathbf{A}$-metric on $\mathcal{E}^{\uparrow}$.

So, given the usual E-H action (without a cosmological constant)

$$\mathfrak{EH}(\mathcal{D}) = \int \mathcal{R}(\mathcal{D})\varpi \tag{69}$$

with ϖ the volume element associated with ρ,[62] (67) reads

$$\frac{d}{dt}(\mathfrak{EH}(\mathcal{D}_t))|_{t=0} \equiv \overbrace{\mathfrak{EH}(\mathcal{D}_t)}^{\cdot}(0) = \int \frac{d}{dt}(\mathcal{R}\varpi)|_{t=0} \tag{70}$$

By setting $\overbrace{\mathfrak{EH}(\mathcal{D}_t)}^{\cdot}(0)$ in (70) equal to zero, one arrives at the vacuum Einstein equations (53) for Lorentzian gravity.

3.3.1 A brief note on the topology of $\mathsf{A}_{\mathbf{A}}(\mathcal{E}^{\uparrow})$

In the introduction we alluded to the general fact that the space of connections is non-linear (*ie*, it is not a vector space) with a 'complicated' topology. Below we would like to comment briefly on the issue of the topology of the space $\mathsf{A}_{\mathbf{A}}(\mathcal{E}^{\uparrow})$ of spin-Lorentzian connections on $\mathcal{E}^{\uparrow}$. This issue is of relevance here since one would like to make sense of the $\frac{d}{dt}$-differentiation of $\mathfrak{EH}$ in (67). Thus, in connection with (67), the crucial question appears to be:

> *with respect to what topology* (on $\mathsf{A}_{\mathbf{A}}(\mathcal{E}^{\uparrow})$) *does one take the limit so as to define the ('variational') derivative of* $\mathfrak{EH}$ *with respect to t (ie, with respect to* $\mathcal{D}$*) in* (67)?[63]

ADG answers this question by first translating it to an equivalent question about convergence in the structure sheaf **A**. That is to say,

can one define limits and convergence in the sheaf **A** *of coefficients?*

To see that this translation is effective, one should realize that *in order to define the derivative of* $\mathfrak{EH}$ *one need only be able to take limits and study convergence in the space where the latter takes values, which, according to* (65), *is* $\mathbf{A}(X)$! Thus, ADG has given so far the following two answers to the question when $\overbrace{\mathfrak{EH}(\gamma)}^{\cdot}$ is well defined:

1. When **A** is a topological algebra sheaf [67, 68, 70, 71].

2. When **A** is Rosinger's algebra of generalized functions [70, 71].

For in both cases **A** has a well defined topology and the related notion of convergence.

In section 5, where we give a finitary, causal and quantal version of the vacuum Einstein equations for Lorentzian gravity (53)—them too derived from a variation of a reticular E-H

[62] We will return to define ϖ shortly.

[63] This question would also be of relevance if for instance one asked whether the map (path) γ in (68) is continuous.

action functional $\overrightarrow{\mathfrak{EH}}_i$, we will give a third example of algebra sheaves—the finsheaves of incidence algebras—in which the notions of convergence, limits and topology (the so-called Rota topology) are well defined so as to 'justify' the corresponding differentiation (variation) $\overbrace{\overrightarrow{\mathfrak{EH}}}^{\cdot}{}_i$.

The discussion above prompts us to make the following clarification:

> *to 'justify' the derivation of Einstein's equations from varying $\mathfrak{EH}$ with respect to $\mathcal{D}$, one need not study the topology of $\mathsf{A}_{\mathbf{A}}(\mathcal{E}^{\uparrow})$ per se. Rather, all that one has to secure is that there is a well defined notion of (local) convergence in $\mathbf{A}$.*[64]

This is how ADG essentially evades the problem of dealing directly with the 'complicated' topology of $\mathsf{A}_{\mathbf{A}}(\mathcal{E}^{\uparrow})$.

We conclude this discussion of the E-H action functional $\mathfrak{EH}$ and its variation yielding the vacuum gravitational equations, by giving a concise ADG-theoretic statement about the *(gauge) invariance of the first* which in turn amounts to the (gauge) covariance of the second. Let $\mathcal{E}^{\uparrow} = (\mathcal{E}, \rho)$ be our usual (real) E-L vector sheaf (of rank 4) and $\mathcal{D}$ a spin-Lorentzian gravitational metric connection on it whose curvature $\mathcal{R}$ is involved in $\mathfrak{EH}(\mathcal{D})$ above. Then,

> *the Einstein-Hilbert functional $\mathfrak{EH}$ is invariant under the action of a (local) ρ-preserving gauge transformation, by which we mean a (local) element (ie, local section) of the structure group sheaf $\mathcal{A}ut_{\mathbf{A}}\mathcal{E}^{\uparrow} \equiv \mathfrak{L}^{+} := \mathcal{A}ut_{\rho}\mathcal{E}$ of $\mathcal{E}^{\uparrow} = (\mathcal{E}, \rho)$, which, in turn, is a subsheaf of $\mathcal{A}ut_{\mathbf{A}}\mathcal{E}$,* where locally, $\mathcal{A}ut_{\mathbf{A}}\mathcal{E}(U) = \mathrm{GL}(4, \mathbf{A}(U)) = \mathcal{GL}(4, \mathbf{A})(U)$.

3.3.2 A brief note on ϖ, the Hodge-$\star$ operator, and on self-duality in ADG

Below, we discuss briefly *à la* ADG the volume element or measure ϖ appearing in the E-H action integral (69), as well as the Hodge-$\star$ operator and the self-dual Lorentzian connections $\mathcal{A}^{+}$ associated with it, thus prepare the ground for a brief comparison we are going to make subsequently between our locally finite, causal and quantal vacuum Einstein gravity and an approach to non-perturbative canonical quantum gravity based on Ashtekar's new variables [3].

[64]This is another example of the general working philosophy of ADG according to which the underlying space or 'domain' so to speak (here $\mathsf{A}_{\mathbf{A}}(\mathcal{E}^{\uparrow})$) is of secondary importance for studying 'differentiability'. For the latter, what is of primary importance is the algebraic structure of the objects that live on that domain. For the notion of derivative, and differentiability in general, one should care more about the structure of the 'target space' or 'range' (here the structure sheaf space $\mathbf{A}$) than that of the 'source space' or 'domain' (here the base space X)—after all, the generic base 'localization' space X employed by ADG is assumed to be just a topological space without having been assigned *a priori* any sort of differential structure whatsoever. Of course, *in the classical case, X is completely characterized, as a differential manifold, by the corresponding structure sheaf $\mathbf{A}_X \equiv \mathcal{C}^{\infty}_X$ of infinitely differentiable (smooth) functions* (in particular, see our comments on Gel'fand duality in 5.5.1). In other words, the classical differential geometric notions 'differential (*ie*, $\mathcal{C}^{\infty}$-smooth) manifold' and 'the topological algebra $\mathcal{C}^{\infty}(X)$' are tautosemous (*ie*, semantically equivalent) notions. Alas, other more general kinds of differentiability, may come from algebraic structures A other than $\mathcal{C}^{\infty}(X)$ that one may localize sheaf-theoretically (as structure sheaves $\mathbf{A}_X$) on an arbitrary topological space X. This is the very essence of ADG and will recur time and again in the sequel.

1. *Volume element.* Let $(X, \mathbf{A})$ be our usual $\mathbf{K}$-algebraized space and $\mathcal{E}$ a free $\mathbf{A}$-module of finite rank n over X, which is locally isomorphic to the 'standard' one $\mathbf{A}^n$. Let also ρ be a strongly non-degenerate (and indefinite, in our case of interest) metric on $\mathcal{E}$, which makes it a *pseudo-Riemannian free* $\mathbf{A}$*-module of finite rank* n *over* X. Then, one considers the sequence $\epsilon \equiv (\epsilon_i)_{1\leq i\leq n}$ of global sections of $\mathcal{E} \simeq \mathbf{A}^n$ (*ie*, $\epsilon_i \in \mathbf{A}^n(X) = \mathbf{A}(X)^n$)—the so-called *Kronecker gauge of* $\mathbf{A}^n$.[65] Then, the volume element ϖ associated with the given $\mathbf{A}$-metric ρ is defined to be

$$\varpi := \sqrt{|\rho|}\epsilon_1 \wedge \cdots \wedge \epsilon_n \in (\wedge^n \mathbf{A}^n)(X) \equiv (\det \mathbf{A}^n)(X) = \mathbf{A}(X) \tag{71}$$

That is to say,

> *the volume element* ϖ *is a nowhere vanishing (because* ρ *is non-degenerate) global section of the structure sheaf* $\mathbf{A}$*. Moreover, since* $(\wedge^n\mathbf{A}^n)^*(X) = (\wedge^n(\mathbf{A}^n)^*)(X) = (\det \mathbf{A}^n)^*(X) = \mathbf{A}(X)$*,* ϖ *can be viewed as an* $\mathbf{A}(X)$*-linear morphism on* $\det(\mathbf{A}^n)$ and, as such, as *a map of* $\mathbf{A}$ *into itself*: $\varpi \in (\mathcal{E}nd\mathbf{A})(X) = \mathrm{End}\mathbf{A} = \mathbf{A}(X)$.

The crux of the argument here is that the definition (71) of ϖ readily applies to the case where X is an Einstein space and $(\mathcal{E}^{\uparrow}, \rho)$ our usual (real) Lorentzian vector sheaf on it. This is so because, as mentioned earlier, $\mathcal{E}^{\uparrow}$ is a locally free $\mathbf{A}$-module of rank 4, that is, locally (*ie*, U-wise) in X: $\mathcal{E}^{\uparrow} \simeq \mathbf{A}^4$. Hence, the volume element ϖ appearing in (69) is now an element of $\mathbf{A}(U)$. Of course, since, by definition, $\mathbf{A}$ is a *fine sheaf*, here too ϖ can be promoted to a global section of $\mathbf{A}$ ($\varpi \in \mathbf{A}(X)$).

2. *Hodge-$\star$.* As with the volume element ϖ, let $(\mathcal{E}, \rho)$ be a pseudo-Riemannian (Lorentzian) free $\mathbf{A}$-module of rank n and recall from (12) the canonical $\mathbf{A}$-isomorphism $\tilde{\rho}$ between the $\mathbf{A}$-modules $\mathcal{E}$ and its dual $\mathcal{E}^*$ induced by ρ. That is to say, $\mathcal{E} \overset{\tilde{\rho}}{\cong} \mathcal{E}^* \equiv \mathcal{H}om_{\mathbf{A}}(\mathcal{E}, \mathbf{A})$.
   We define the following $\mathbf{A}$-isomorphism $\star$ of $\mathbf{A}$-modules

$$\star : \quad \wedge^p \mathcal{E}^* \longrightarrow \wedge^{n-p}\mathcal{E}^* \tag{72}$$

To give $\star$'s section-wise action, we need to define first, for any $v \in \wedge^{n-p}\mathcal{E}(X)$,

$$v^{\circledast} := (\wedge^{n-p}\tilde{\rho})(v) \in \wedge^{n-p}\mathcal{E}^*(X) = (\wedge^{n-p}\mathcal{E}(X))^* \tag{73}$$

so that then we can define

$$(\star u)(v) := \varpi(u \wedge v^{\circledast}) \equiv (u \wedge v^{\circledast}) \cdot \varpi \in \mathbf{A}(X) \tag{74}$$

for $u \in \wedge^p\mathcal{E}^*(X) = \wedge^p\mathcal{E}(X)^*$.

[65] In ADG, this appellation for ϵ is reserved for positive definite (Riemannian) metrics ρ [67], but here we extend the nomenclature to include indefinite metrics as well.

Two things can be mentioned at this point: first, that for the *identity or unit global section* $\mathbf{1}$ *of* $\mathbf{A}$, $\star\mathbf{1} = \varpi$, and second, that $\star$ entails an $\mathbf{A}$-isomorphism of the $\mathbf{A}$-module defined by the exterior algebra of $\mathcal{E}^*$, $\wedge\mathcal{E}^*$, into itself. The latter means, in turn, that $\star$ is an element of $\mathcal{A}ut_{\mathbf{A}}(\wedge\mathcal{E}^*)$.

> *The map* $\star$ *of* (72) *and* (74) *is the ADG-theoretic version of the usual Hodge-*$\star$ *operator induced by the* $\mathbf{A}$*-metric* ρ.

3. *Self-dual Lorentzian connections* $\mathcal{A}^+$. Now that we have $\star$ at our disposal, we can define a particular class of Y-M $\mathbf{A}$-connections $\mathcal{D}^+$ on vector sheaves, the so-called *self-dual connections*, whose gauge potential parts $\mathcal{A}^+$ are coined *self-dual gauge fields*. So, we let $(\mathcal{E}, \rho, \mathcal{D})$ be a L-Y-M field on a L-Y-M space X. The definition of $\mathcal{D}^+$s pertains to the property that their curvatures, $\mathcal{F}^+ := \mathcal{F}(\mathcal{D}^+)$, satisfy relative to the Hodge-$\star$ duality operator

$$\star\mathcal{F}^+ = \mathcal{F}^+ \tag{75}$$

hence their name *self-dual*.

In view of (75) and the second Bianchi identity (49), we have

$$\begin{aligned} \delta^2_{\mathcal{E}nd\mathcal{E}}(\mathcal{F}^+) = ((-1)^{n\cdot 3+1}\star\mathcal{D}^{n-2}\star)(\mathcal{F}^+) = (-1)^{1+3n}\star\mathcal{D}^{n-2}(\mathcal{F}^+) = \\ = (-1)^{1+3n}\star\mathcal{D}^2_{\mathcal{E}nd\mathcal{E}}(\mathcal{F}^+) = 0 \end{aligned} \tag{76}$$

the point being that the (field strengths $\mathcal{F}^+$ of the) self-dual connections $\mathcal{D}^+$ also satisfy the Y-M equations. We will return to self-dual connections in section 5 where we will discuss the close affinity between our finitary, causal and quantal version of vacuum Einstein Lorentzian gravity and a recent approach to non-perturbative quantum gravity which uses Ashtekar's new (canonical) variables [3].

3.4 Y-M and Gravitational Moduli Space: $\mathcal{G}$-Equivalent Connections

In the present subsection we will give a short account of the ADG-theoretic perspective on moduli spaces of L-Y-M connections, focusing our attention on the corresponding moduli spaces of spin-Lorentzian (vacuum) gravitational connections that are of special interest to our investigations in this paper.

To initiate our presentation, we consider a (real) Lorentzian vector sheaf $\mathcal{E}^\uparrow = (\mathcal{E}, \rho)$ and we recall from 2.4 the affine space $\mathsf{A}_{\mathbf{A}}(\mathcal{E})$ of metric $\mathbf{A}$-connections on it (54). From our discussion of $\mathcal{G}$-sheaves in 2.2, we further suppose that $\mathcal{E}^\uparrow$ is the associated sheaf of the principal sheaf $\mathfrak{L}^+ := \mathcal{A}ut_{\mathbf{A}}\mathcal{E}^\uparrow \equiv \mathcal{A}ut_\rho\mathcal{E}$—the group sheaf of ρ-preserving $\mathbf{A}$-automorphisms of $\mathcal{E}$ (the structure group sheaf of $\mathcal{E}^\uparrow$, which is also the (local) invariance group of the free Y-M action functional $\mathfrak{YM}(\mathcal{D})$ [67, 68]).[66] Our main contention in this section is that

[66]In the case of the functional $\mathfrak{EH}(\mathcal{D})$ on $(\mathcal{E}^\uparrow, \mathcal{D})$, we saw in the previous subsection that its (local) invariance (structure) group is precisely $(\mathrm{Aut}_{\mathbf{A}}\mathcal{E}^\uparrow)(U) := \Gamma(U, \mathcal{A}ut_{\mathbf{A}}\mathcal{E}^\uparrow) \equiv (\mathrm{Aut}_\rho\mathcal{E})(U) =: \mathfrak{L}^+(U) \simeq L^\uparrow$.

the (global) gauge group $\mathcal{A}ut_{\mathbf{A}}\mathcal{E}^{\uparrow}(X) \equiv \mathrm{Aut}_{\mathbf{A}}\mathcal{E}^{\uparrow} \equiv \mathfrak{L}^{+}(X) := \mathrm{Aut}_{\rho}\mathcal{E}$ *acts on the affine space* $\mathsf{A}_{\mathbf{A}}(\mathcal{E}^{\uparrow})$ *of metric* $\mathbf{A}$*-connections on the Lorentzian vector sheaf* $\mathcal{E}^{\uparrow} = (\mathcal{E}, \rho)$.

Let us elaborate a bit on the statement above, which will subsequently lead us to define moduli spaces of gauge-equivalent connections.

We have already alluded to the fact, in connection with the (local) transformation law of gauge potentials $\mathcal{A}$ of $\mathbf{A}$-connections $\mathcal{D}$ on general vector sheaves $\mathcal{E}$ at the end of 2.1, that one may be able to establish an equivalence relation $\mathcal{A} \overset{g}{\sim} \mathcal{A}'$ between them, for g a local gauge transformation (*ie*, a local section of the structure $\mathcal{G}$-sheaf $\mathcal{A}ut_{\mathbf{A}}(\mathcal{E})$ of $\mathcal{E}$; $g \in \mathcal{A}ut_{\mathbf{A}}(\mathcal{E})(U) = \mathcal{GL}(n, \mathbf{A})(U)$). We can extend this equivalence relation from the gauge potentials $\mathcal{A}$ to their full connections $\mathcal{D}$, as follows.

Schematically, and in general, for an $\mathbf{A}$-module $\mathcal{E}$ we say that two connections $\mathcal{D}$ and $\mathcal{D}^{'}$ on it are gauge-equivalent if there exists an element $g \in \mathrm{Aut}(\mathcal{E})$ making the following diagram commutative

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\mathcal{D}"] \arrow[d, "g"'] & \mathbf{\Omega}(\mathcal{E}) \arrow[d, "g \otimes \mathbf{1}_{\mathbf{\Omega}} \equiv g \otimes \mathbf{1}"] \\ \mathcal{E} \arrow[r, "\mathcal{D}^{'}"'] & \mathbf{\Omega}(\mathcal{E}) \end{tikzcd} \tag{77}$$

which is read as

$$\mathcal{D}^{'} \circ g = (g \otimes \mathbf{1}) \circ \mathcal{D} \Leftrightarrow \mathcal{D}^{'} = (g \otimes \mathbf{1}) \circ \mathcal{D} \circ g^{-1} \tag{78}$$

or in terms of the adjoint representation $\mathrm{Ad}(\mathcal{G})$ of the structure group $\mathcal{G} \ni g$

$$\mathcal{D}^{'} = g \circ \mathcal{D} \circ g^{-1} \equiv g\mathcal{D}g^{-1} =: \mathrm{Ad}(g)\mathcal{D} \tag{79}$$

It is now clear that

(78) and (79) *define an equivalence relation* $\overset{g}{\sim}$ *on* $\mathsf{A}_{\mathbf{A}}(\mathcal{E})$: $\mathcal{D} \overset{g}{\sim} \mathcal{D}'$, $g \in \mathrm{Aut}\mathcal{E}$. $\overset{g}{\sim}$ *is precisely the equivalence relation defined by the action of the structure group* $\mathrm{Aut}\mathcal{E}$ *of* $\mathcal{E}$ *on* $\mathsf{A}_{\mathbf{A}}(\mathcal{E})$, *as alluded to above.*

Thus, it is natural to consider the following $\mathcal{G}$-action α on $\mathsf{A}_{\mathbf{A}}(\mathcal{E})$

$$\alpha : \ \mathrm{Aut}\mathcal{E} \times \mathsf{A}_{\mathbf{A}}(\mathcal{E}) \longrightarrow \mathsf{A}_{\mathbf{A}}(\mathcal{E}) \tag{80}$$

defined point-wise by

$$(g, \mathcal{D}) \mapsto \alpha(g, \mathcal{D}) \equiv g \cdot \mathcal{D} \equiv g(\mathcal{D}) := g\mathcal{D}g^{-1} \equiv \mathrm{Ad}(g)\mathcal{D} \tag{81}$$

with the straightforward identification from (78)

$$g(\mathcal{D}) \equiv g\mathcal{D}g^{-1} \equiv (g \otimes 1) \circ \mathcal{D} \circ g^{-1} \in \mathrm{Hom}_{\mathbf{C}}(\mathcal{E}, \mathbf{\Omega}(\mathcal{E})) \tag{82}$$

In turn, for a given $\mathcal{D} \in \mathsf{A}_{\mathbf{A}}(\mathcal{E})$, α delimits the following set in $\mathsf{A}_{\mathbf{A}}(\mathcal{E})$

$$\begin{aligned} \mathcal{O}_{\mathcal{D}} &:= \{g \cdot \mathcal{D} \in \mathsf{A}_{\mathbf{A}}(\mathcal{E}) : \ g \in \mathrm{Aut}\mathcal{E}\} = \\ &= \{\mathcal{D}' \in \mathsf{A}_{\mathbf{A}}(\mathcal{E}) : \ \mathcal{D}' \overset{g}{\sim} \mathcal{D}, \text{for some } g \in \mathrm{Aut}\mathcal{E}\} \end{aligned} \tag{83}$$

called *the orbit of an* $\mathbf{A}$*-connection* $\mathcal{D}$ *on* $\mathcal{E}$ *under the action* α *of the gauge group* $\mathcal{G} = \mathrm{Aut}\mathcal{E}$ *on* $\mathsf{A}_{\mathbf{A}}(\mathcal{E})$. $\mathcal{O}_{\mathcal{D}}$ consists of all connections $\mathcal{D}'$ in $\mathsf{A}_{\mathbf{A}}(\mathcal{E})$ that are gauge-equivalent to $\mathcal{D}$.

Following [67, 68], we would also like to note that it can be shown that the gauge-orbit $\mathcal{O}_{\mathcal{D}}$ in (83) can be equivalently written in terms of the induced connection $\mathcal{D}_{\mathcal{E}nd\mathcal{E}}$ as follows

$$\mathcal{O}_{\mathcal{D}} = \{\mathcal{D} - \mathcal{D}_{\mathcal{E}nd\mathcal{E}}(g)g^{-1} : \ g \in \mathrm{Aut}\mathcal{E}\} \tag{84}$$

At the same time, the *stability group* $\mathcal{O}(\mathcal{D})$ of $\mathcal{D} \in \mathsf{A}_{\mathbf{A}}(\mathcal{E})$ under the action of $\mathrm{Aut}(\mathcal{E})$ is, by definition, the set of all $g \in \mathrm{Aut}\mathcal{E}$ such that $g \cdot \mathcal{D} = \mathcal{D}$, so that

$$\begin{aligned} \mathcal{O}(\mathcal{D}) = \ker(\mathcal{D}_{\mathcal{E}nd\mathcal{E}|\mathrm{Aut}\mathcal{E}}) \equiv \{g \in \mathrm{Aut}\mathcal{E} : \ \mathcal{D}_{\mathcal{E}nd\mathcal{E}}(g) = 0\} = \\ \{g \in \mathrm{Aut}\mathcal{E} : \ [\mathcal{D}, g] := \mathcal{D}g - g\mathcal{D} = 0\} \end{aligned} \tag{85}$$

which means that the stability group of the connection $\mathcal{D} \in \mathsf{A}_{\mathbf{A}}(\mathcal{E})$ consists of all those (gauge) transformations of $\mathcal{E}$ ($g \in \mathrm{Aut}\mathcal{E}$) that commute with $\mathcal{D}$.

At this point, and before we define moduli spaces of gauge-equivalent connections ADG-theoretically, we would like to digress a bit and make a few comments on the possibility of developing differential geometric ideas (albeit, not of a classical, geometrical $\mathcal{C}^\infty$-smooth sort, but of an algebraic ADG kind) on the affine space $\mathsf{A}_{\mathbf{A}}(\mathcal{E})$. The remarks below are expressed in order to prepare the reader for comments on the possibility of developing differential geometry on the gauge moduli space of gravitational connections that we are going to make in 5.3 in connection with some problems (*eg*, Gribov's ambiguity) people have encountered in trying to quantize general relativity (regarded as a gauge theory) both canonically (*ie*, in a Hamiltonian fashion) and covariantly (*ie*, in a Lagrangian fashion). It is exactly due to these problems that others have also similarly felt the need of developing differential geometric concepts and constructions (albeit, of the classical, $\mathcal{C}^\infty$-sort) on moduli spaces of Y-M and gravitational connections [7, 8].

As a first differential geometric idea on $\mathsf{A}_{\mathbf{A}}(\mathcal{E})$, we would first like to define a set of objects (to be regarded as abstract 'tangent vectors') that would qualify as the *'tangent space' of* $\mathsf{A}_{\mathbf{A}}(\mathcal{E})$ *at any of its points* $\mathcal{D}$, and then, after we define moduli spaces of gauge-equivalent connections below, we would also like to define an analogous *'tangent space' to the moduli space at a gauge-orbit* $\mathcal{O}_{\mathcal{D}}$ *of a connection* $\mathcal{D} \in \mathsf{A}_{\mathbf{A}}(\mathcal{E})$.

We saw earlier (2.4) that for $\mathcal{E}$ a vector sheaf of rank n, the affine space $\mathsf{A}_{\mathbf{A}}(\mathcal{E})$ can be modelled after $\mathbf{\Omega}^1(\mathcal{E}nd\mathcal{E})(X)$. *We actually define the latter space to be the sought after 'tangent space' of* $\mathsf{A}_{\mathbf{A}}(\mathcal{E})$ *at any of its 'points'* $\mathcal{D}$. That is to say,

$$T(\mathsf{A}_{\mathbf{A}}(\mathcal{E}), \mathcal{D}) := \mathbf{\Omega}^1(\mathcal{E}nd\mathcal{E})(X) \tag{86}$$

and we recall from the foregoing that $\mathbf{\Omega}^1(\mathcal{E}nd\mathcal{E})(X)$ is itself an $\mathbf{A}(X)$-module which locally, relative to a gauge U, becomes the $n \times n$-matrix of 1-forms $\mathbf{A}(U)$-module $M_n(\mathbf{\Omega}^1(U)) = M_n(\mathbf{\Omega}^1)(U)$.[67]

[67] As a matter of fact, one can actually prove (86) along classical lines—for example, by fixing a point $\mathcal{D}$ in

We are now in a position to define the *global moduli space or gauge orbit space of the* **A**-*connections on* $\mathcal{E}$, as follows

$$M(\mathcal{E}) \equiv \mathsf{A}_{\mathbf{A}}(\mathcal{E})/\mathrm{Aut}\mathcal{E} := \bigcup_{\mathcal{D}\in\mathsf{A}_{\mathbf{A}}(\mathcal{E})} \mathcal{O}_{\mathcal{D}} = \sum_{\mathcal{D}} \mathcal{O}_{\mathcal{D}} \tag{87}$$

The epithet 'global' above indicates that the quotient in (87) can be actuallly localized—something that comes in handy when one, as we do, works with a vector sheaf $\mathcal{E}$ on X and the latter is gauged relative to a local frame $\mathcal{U} = \{U\}$. The localization of $M(\mathcal{E})$ means essentially that one uses the *sheaf of germs of moduli spaces of the* **A**-*connections of the module or vector sheaf* $\mathcal{E}$ *in focus*. To see this, the reader must realize that, as U ranges over the open subsets of X, one deals with a (complete) presheaf of orbit spaces equipped with the obvious restriction maps. To follow this line of thought, one first observes the inclusion

$$\mathsf{A}_{\mathbf{A}}(\mathcal{E})|_U \subseteq \mathsf{A}_{\mathbf{A}|_U}(\mathcal{E}_U) \tag{88}$$

and a similar restriction of the structure group sheaf $\mathcal{G} \equiv \mathcal{A}ut\mathcal{E}$. Then, section-wise over U one has

$$\begin{gathered}(\mathcal{A}ut\mathcal{E})|_U = (\mathcal{A}ut\mathcal{E})(U) = \mathrm{Isom}_{\mathbf{A}|_U}(\mathcal{E}|_U, \mathcal{E}|_U) = \\ = \mathcal{I}som_{\mathbf{A}|_U}(\mathcal{E}|_U, \mathcal{E}|_U)(U) \equiv \mathcal{A}ut(\mathcal{E}|_U)(U) = \mathrm{Aut}(\mathcal{E}|_U)\end{gathered} \tag{89}$$

thus, *in toto*, the following local equality

$$\mathcal{A}ut\mathcal{E}(U) = \mathrm{Aut}(\mathcal{E}|_U) \tag{90}$$

for every open U in X.

So, in complete analogy to (81), one has the action of $\mathrm{Aut}(\mathcal{E}|_U)$ on the local sets $\mathsf{A}_{\mathbf{A}}(\mathcal{E})|_U$ of **A**-connections in (88)

$$\mathrm{Aut}(\mathcal{E}|_U) \times \mathsf{A}_{\mathbf{A}}(\mathcal{E})|_U \longrightarrow \mathsf{A}_{\mathbf{A}}(\mathcal{E})|_U \tag{91}$$

entailing the following 'orbifold sheaf' of gauge-equivalent **A**-connections on $\mathcal{E}$

$$\mathcal{M}(\mathcal{E}) = \mathsf{A}_{\mathbf{A}}(\mathcal{E})/\mathcal{A}ut\mathcal{E} \tag{92}$$

$\mathcal{M}(\mathcal{E})$ is the aforesaid sheaf of germs of moduli spaces of **A**-connections on $\mathcal{E}$.

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the affine space $\mathsf{A}_{\mathbf{A}}(\mathcal{E})$, regard it as 'origin' (*ie*, the zero vector 0), let a curve $\gamma(t)$ in $\mathsf{A}_{\mathbf{A}}(\mathcal{E})$ pass through it (*ie*, $\mathcal{D} \equiv \mathcal{D}_0 = \gamma(0)$), and then find the vector $\dot{\gamma}(t)$ tangent to γ. This proof has been shown to work in the particular case the structure sheaf **A** is a topological vector space sheaf [68, 71] (and in section 5 we will see that it also works in the case of our finsheaves of incidence algebras for deriving the locally finite, causal and quantal vacuum Einstein equations for Lorentzian gravity); in fact, we used it in (67) and (68) to derive the vacuum Einstein equations from a variational principle on the space of Lorentzian connections.

Finally, it must also be mentioned here, in connection with the local isomorphism $\mathcal{E} \simeq \mathbf{A}^n$ of a vector sheaf $\mathcal{E}$ mentioned earlier, that $(\mathcal{A}ut_{\mathbf{A}}\mathcal{E})(U)$ above reduces locally to $\mathcal{GL}(n,\mathbf{A})(U) = \mathrm{GL}(n,\mathbf{A}(U))$, as follows[68]

$$\begin{aligned}(\mathcal{A}ut_{\mathbf{A}}\mathcal{E})(U) = \mathrm{Aut}(\mathcal{E}|_U) = \mathcal{A}ut(\mathbf{A}^n|_U) = (\mathcal{A}ut\mathbf{A}^n)(U) = \\ = M_n(\mathbf{A})^{\bullet}(U) \equiv \mathcal{GL}(n,\mathbf{A})(U) \equiv \mathrm{GL}(n,\mathbf{A}(U))\end{aligned} \tag{93}$$

We can distill this to the following remark

> *any local automorphism of a given vector sheaf $\mathcal{E}$ of rank n over one of its local gauges U is effectively given by a local automorphism of $\mathbf{A}^n$—that is to say, by an element of* $\mathrm{GL}(n,\mathbf{A}(U)) = \mathcal{GL}(n,\mathbf{A})(U) \equiv \mathrm{GL}(n,\mathbf{A}|_U)$.

so that the gauge (structure) group $\mathcal{A}ut_{\mathbf{A}}\mathcal{E}$ of $\mathcal{E}$ is locally (*ie*, U-wise) reduced to the group sheaf $\mathcal{GL}(n,\mathbf{A})$,[69] as it has been already anticipated, for example, in 2.1.2 in connection with the transformation law of gauge potentials,[70] and earlier in connection with vacuum Einstein Lorentzian gravity on $\mathcal{E}^{\uparrow}$.

As noted before, now that we have defined moduli spaces of gauge-equivalent connections, and similarly to the 'tangent space' $T(\mathsf{A}_{\mathbf{A}}(\mathcal{E}),\mathcal{D})$ in (86), we would like to define $T(\mathcal{O}_{\mathcal{D}},\mathcal{D})$—*the 'tangent space' to a gauge-orbit of an element $\mathcal{D} \in \mathsf{A}_{\mathbf{A}}(\mathcal{E})$* and, *in extenso*, $T(M(\mathcal{E}),\mathcal{O}_{\mathcal{D}})$—*the 'tangent space' to the moduli space of $\mathcal{E}$ at an orbit of $\mathcal{D} \in \mathsf{A}_{\mathbf{A}}(\mathcal{E})$*. We have seen how the induced $\mathbf{A}$-connection of the vector sheaf $\mathcal{E}nd\mathcal{E}$

$$\mathcal{D}_{\mathcal{E}nd\mathcal{E}} :\ \mathcal{E}nd\mathcal{E} \longrightarrow \mathbf{\Omega}^1(\mathcal{E}nd\mathcal{E}) \tag{94}$$

can be viewed as the 'covariant differential' of the connection $\mathcal{D}$ in $\mathsf{A}_{\mathbf{A}}(\mathcal{E})$. By defining the induced coderivative $\delta^1_{\mathcal{E}nd\mathcal{E}}$ adjoint to $\mathcal{D}_{\mathcal{E}nd\mathcal{E}}$ as

$$\delta^1_{\mathcal{E}nd\mathcal{E}} :\ \mathbf{\Omega}^1(\mathcal{E}nd\mathcal{E}) \longrightarrow \mathcal{E}nd\mathcal{E} \tag{95}$$

we define

$$\mathcal{S}_{\mathcal{D}} := \mathcal{D} + \ker\delta^1_{\mathcal{E}nd\mathcal{E}} \equiv \{\mathcal{D} + u \in \mathsf{A}_{\mathbf{A}}(\mathcal{E}) :\ \delta^1_{\mathcal{E}nd\mathcal{E}}(u) = 0\} \tag{96}$$

for $u \in \mathbf{\Omega}^1(\mathcal{E}nd\mathcal{E})(X)$. Of course, for $u = 0 \in \mathbf{\Omega}^1(\mathcal{E}nd\mathcal{E})(X)$, one sees that $\mathcal{D}$ belongs to $\mathcal{S}_{\mathcal{D}}$, so that

> *$\mathcal{S}_{\mathcal{D}}$ is a subspace of $\mathsf{A}_{\mathbf{A}}(\mathcal{E})$ through $\mathcal{D}$.* In fact, one can show [68, 71] that *$\mathcal{S}_{\mathcal{D}}$ is an affine $\mathbf{C}$-linear subspace of $\mathsf{A}_{\mathbf{A}}(\mathcal{E})$ through the point $\mathcal{D}$, modelled after* $(\ker\delta^1_{\mathcal{E}nd\mathcal{E}})(X)$.[71]

[68] In the case of $\mathcal{E}^{\uparrow}$, the local reduction below has already been anticipated earlier.

[69] Or equivalently, to its complete presheaf of sections $\Gamma(\mathcal{GL}(n,\mathbf{A}))$.

[70] See remarks after (9).

[71] $(\ker\delta^1_{\mathcal{E}nd\mathcal{E}})(X)$ being in fact a sub-$\mathbf{A}(X)$-module of $\mathbf{\Omega}^1(\mathcal{E}nd\mathcal{E})(X)$.

Moreover, and this is crucial for defining $T(\mathcal{O}_{\mathcal{D}}, \mathcal{D})$, one is able to prove [68, 71] that

$$\mathrm{im}\mathcal{D}_{\mathcal{E}nd\mathcal{E}} \oplus \ker\delta^1_{\mathcal{E}nd\mathcal{E}} = \mathbf{\Omega}^1(\mathcal{E}nd\mathcal{E})(X) =: T(\mathsf{A}_{\mathbf{A}}(\mathcal{E}), \mathcal{D}) \tag{97}$$

for any local gauge U of $\mathcal{E}$.

In toto, since both $\mathcal{D}_{\mathcal{E}nd\mathcal{E}}$ and $\delta^1_{\mathcal{E}nd\mathcal{E}}$ are restricted on the gauge group $\mathrm{Aut}\mathcal{E}$, and in view of (84), one realizes that

$$T(\mathcal{O}_{\mathcal{D}}, \mathcal{D}) = \mathrm{im}(\mathcal{D}_{\mathcal{E}nd\mathcal{E}|\mathrm{Aut}\mathcal{E}}) = \ker(\delta^1_{\mathcal{E}nd\mathcal{E}|\mathrm{Aut}\mathcal{E}})^{\perp} \tag{98}$$

where '$\perp$' designates 'orthogonal subspace' with respect to the $\mathbf{A}$-metric ρ on $\mathcal{E}$. Thus,

> $\mathcal{S}_{\mathcal{D}}$ *is the orthogonal complement of the tangent space* $T(\mathcal{O}_{\mathcal{D}}, \mathcal{D})$ *to the orbit* $\mathcal{O}_{\mathcal{D}}$ *of* $\mathcal{D}$ *at the point* $\mathcal{D}$ *of* $\mathsf{A}_{\mathbf{A}}(\mathcal{E})$.

At the same time, for 'infinitesimal variations' $u \in \mathbf{\Omega}^1(\mathcal{E}nd\mathcal{E})(X)$ around $\mathcal{D} \in \mathsf{A}_{\mathbf{A}}(\mathcal{E})$, one can show [68, 71]

$$\begin{gathered} T(\mathcal{O}_{\mathcal{D}+u}, \mathcal{D}+u) = \mathrm{im}((\mathcal{D}+u)_{\mathcal{E}nd\mathcal{E}|\mathrm{Aut}\mathcal{E}}) = \\ \mathrm{im}((\mathcal{D}_{\mathcal{E}nd\mathcal{E}}+u)|_{\mathrm{Aut}\mathcal{E}}) = \{(\mathcal{D}_{\mathcal{E}nd\mathcal{E}}+u)g :\ g \in \mathrm{Aut}\mathcal{E}\} \end{gathered} \tag{99}$$

Concomitantly, in order to arrive at $T(M(\mathcal{E}), \mathcal{O}_{\mathcal{D}})$, one realizes [68, 71] that

> *the gauge group* $\mathrm{Aut}\mathcal{E}$ *acts on* $\mathsf{A}_{\mathbf{A}}(\mathcal{E})$ *in a way that is compatible with its affine structure.*

That is to say, one has

$$g(\mathcal{D}+u) = g\mathcal{D} + gu,\ \forall g \in \mathrm{Aut}\mathcal{E}\,\mathrm{and}\,u \in \mathbf{\Omega}^1(\mathcal{E}nd\mathcal{E})(X) \tag{100}$$

The bottom-line of these remarks is that

> $M(\mathcal{E}) := \mathsf{A}_{\mathbf{A}}(\mathcal{E})/\mathrm{Aut}\mathcal{E}$ *can still be construed as an affine space modelled after* $\mathbf{\Omega}^1(\mathcal{E}nd\mathcal{E})(X)/\mathrm{Aut}\mathcal{E} \simeq (\mathrm{im}(\mathcal{D}_{\mathcal{E}nd\mathcal{E}|\mathrm{Aut}\mathcal{E}}))^{\perp} \simeq \mathcal{S}_{\mathcal{D}}$.

Hence one concludes that

$$T(M(\mathcal{E}), \mathcal{O}_{\mathcal{D}}) \simeq \mathcal{S}_{\mathcal{D}} \tag{101}$$

Now that we have $M(\mathcal{E})$ we are in a position to define similarly moduli spaces of (self-dual) spin-Lorentzian connections. Of course, our definition of 'tangent spaces' on $\mathcal{O}_{\mathcal{D}}$ and on $M(\mathcal{E})$ above carries through, virtually unaltered, to the particular (self-dual) Lorentzian case. As noted above, this will become relevant in section 5 where, in view of certain problems that both the canonical and the covariant quantization approaches to quantum general relativity (based on the Ashtekar variables) encounter, the need to develop differential geometric ideas and techniques on the moduli space of (self-dual) spin-Lorentzian connections has arisen in the last decade or so.

3.4.1 Moduli space of (self-dual) spin-Lorentzian connections $\mathcal{D}^{(+)}$

The last remark prompts us to comment briefly on the space of gauge-equivalent (self-dual) spin-Lorentzian connections on the (real) Lorentzian vector sheaf $\mathcal{E}^{\uparrow} = (\mathcal{E}, \rho)$ of rank 4 which is of special interest to us in the present paper. When the latter is endowed with a (self-dual) Lorentzian metric connection $\mathcal{D}^{(+)}$ which (*ie*, whose curvature scalar $\mathcal{R}^{(+)}(\mathcal{D}^{(+)})$) is a solution of (the self-dual version of) (53),[72] it is reasonable to enquire about other gauge-equivalent (self-dual) E-L fields $(\mathcal{E}^{\uparrow}, \breve{\mathcal{D}}^{(+)})$, with $\mathcal{D}^{(+)} \stackrel{g}{\sim} \breve{\mathcal{D}}^{(+)}$ $(g \in \mathcal{G} = \mathcal{A}ut_{\mathbf{A}}\mathcal{E}^{\uparrow})$.

From what has been said above, one readily obtains the local gauge group of $\mathcal{E}^{\uparrow}$

$$\begin{aligned} \mathcal{A}ut_{\mathbf{A}}\mathcal{E}^{\uparrow}(U) \equiv \mathcal{A}ut_{\rho}\mathcal{E}(U) = \mathrm{Aut}_{\rho}(\mathcal{E}|_U) =: \mathfrak{L}^{+}(U) \simeq \\ L^{\uparrow} \subset M_4(\mathbf{A})^{\bullet}(U) = \mathcal{GL}(4, \mathbf{A})(U) = \mathrm{GL}(4, \mathbf{A}(U)) \end{aligned} \tag{102}$$

and, like in (92), we obtain the localized moduli space ('orbifold sheaf') of gauge-equivalent (self-dual) spin-Lorentzian $\mathbf{A}$-connections $\mathcal{D}^{(+)}$ (or their gauge potential parts $\mathcal{A}^{(+)}$) on $\mathcal{E}^{\uparrow}$

$$\mathcal{M}^{(+)}(\mathcal{E}^{\uparrow}) = \mathsf{A}^{(+)}_{\mathbf{A}}(\mathcal{E}^{\uparrow})/\mathcal{A}ut_{\mathbf{A}}\mathcal{E}^{\uparrow} \equiv \mathsf{A}^{(+)}_{\mathbf{A}}(\mathcal{E}^{\uparrow})/\mathcal{A}ut_{\rho}\mathcal{E} \tag{103}$$

Finally, in a possible covariant quantization scenario for vacuum Einstein Lorentzian gravity that we are going to discuss in section 5, $\mathcal{M}(\mathcal{E}^{\uparrow})$ may be regarded as the (quantum) configuration space of the theory in a way analogous to the scheme that has been proposed in the context of Ashtekar's new variables for non-perturbative canonical quantum gravity [3, 6, 7, 8]. In connection with the latter, we note that since the main dynamical variable is a *self-dual* spin-Lorentzian connection $\mathcal{D}^{+}$[73] (see end of 3.3), the corresponding moduli space is denoted by

$$\mathcal{M}^{+}(\mathcal{E}^{\uparrow}) = \mathsf{A}^{+}_{\mathbf{A}}(\mathcal{E}^{\uparrow})/\mathcal{A}ut_{\mathbf{A}}\mathcal{E}^{\uparrow} \equiv \mathsf{A}^{+}_{\mathbf{A}}(\mathcal{E}^{\uparrow})/\mathcal{A}ut_{\rho}\mathcal{E} \tag{104}$$

where, as we have already mentioned earlier, the (local) orthochronous Lorentz structure (gauge) symmetries $\mathcal{G}$ of $\mathcal{E}^{\uparrow}$ can be written as $\mathcal{A}ut_{\mathbf{A}}\mathcal{E}^{\uparrow}(U) \equiv \mathcal{A}ut_{\rho}\mathcal{E}(U) = L^{\uparrow} := SO(1,3)^{\uparrow} \stackrel{\text{locally}}{\simeq} SL(2,\mathbb{C}) \subset M_2(\mathbb{C})$.[74]

4 Kinematics for a Finitary, Causal and Quantal Lorentzian Gravity

One of our main main aims in this paper is to show that the general ADG-theoretic concepts and results presented in the last two sections are readily applicable in the particular case of the *curved finsheaves of qausets* perspective on (the kinematics of) Lorentzian gravity that has been

[72] Which in turn means that $(\mathcal{E}^{\uparrow}, \mathcal{D}^{(+)}) \equiv (\mathcal{E}, \rho, \mathcal{D}^{(+)})$ defines a (self-dual) E-L field.

[73] Or again, locally, its gauge potential part $\mathcal{A}^{+}$.

[74] Always remembering of course that $L^{\uparrow} = SO(1,3)^{\uparrow}$ and its double covering spin-group $SL(2,\mathbb{C})$ are only locally (*ie*, Lie algebra-wise) isomorphic (*ie*, $sl(2,\mathbb{C}) \simeq so(1,3)^{\uparrow}$). Also, for a general (real) Lorentzian vector sheaf $(\mathcal{E}, \rho)$ of rank n, which locally reduces to $\mathbf{A}^n$ (*ie*, it is a locally free $\mathbf{A}$-module), its local (structure) group of Lorentz transformations is $\mathcal{A}ut_{\rho}\mathcal{E}(U) = \mathcal{SL}(n, \mathbf{A})(U) \equiv \mathrm{SL}(n, \mathbf{A}(U)) \subset \mathcal{A}ut_{\mathbf{A}}\mathcal{E}(U) = \mathcal{GL}(n, \mathbf{A})(U) \equiv \mathrm{GL}(n, \mathbf{A}(U)) \equiv M_n(\mathbf{A})^{\bullet}(U) = (\mathcal{E}nd_{\mathbf{A}}\mathcal{E})^{\bullet}(U)$.

developed in the two past papers [74, 75]. In the present section, we recall in some detail from [74], always under the prism of ADG, the main kinematical structures used for a locally finite, causal and quantal version of vacuum Einstein Lorentzian gravity, thus we prepare the ground for the dynamical equations to be described 'finitarily' in the next. In the last subsection (4.3), and with the reader in mind, we give a concise *résumé*—a 'causal finitarity' manual so to speak—of some (mostly new) key kinematical concepts and constructions to be described *en passant* below.

More analytically, we will go as far as to present a finitary version of the (self-dual) moduli space $\mathcal{M}^{(+)}(\mathcal{E}^{\uparrow})$ in (103) and (104) above—arguably, *the* appropriate (quantum) kinematical configuration space for a possible (quantum) theoresis of the (self-dual) spin-Lorentzian connections $\mathcal{A}_i^{(+)}$ inhabiting the aforesaid finsheaves of qausets. We will also present, based on recent results about projective and inductive limits in the category $\mathfrak{DT}$ of Mallios' differential triads [81, 82], as well as on results about projective limits of inverse systems of principal sheaves endowed with Mallios' $\mathbf{A}$-connections [124, 125, 126], the recovery, at the *projective limit* of infinite refinement (or localization) of an *inverse system of principal finsheaves of qausets and reticular spin-Lorentzian connections on them*, of a structure that, from the ADG-theoretic perspective, comes very close to, but does not reproduce exactly, the kinematical structure of classical gravity in its gauge-theoretic guise—the principal orthochronous Lorentzian fiber bundle $\mathcal{P}^{\uparrow}$ over a $\mathcal{C}^{\infty}$-smooth spacetime manifold M endowed with a non-trivial (self-dual) smooth spin-Lorentzian connection $\mathcal{D}^{(+)}$ on it (subsection 4.2).[75] In this way, we are going to be able to make brief comparisons, even if just preliminarily at this early stage of the development of our theory, between a similar differential geometric scheme on the moduli space of gauge-equivalent spin-Lorentzian connections that has been worked out in [8], like ADG, *through entirely algebraic methods*.[76] However, and this must be stressed from the start,

> unlike [8], where projective limit techniques are used in order to endow (a completion of) the moduli space of gauge-equivalent connections with a differential manifold-like structure, thus (be able to) induce to it classical differential geometric notions such as differential forms, exterior derivatives, vector fields, volume forms *etc*, *we, with the help of ADG, already possess those at the finitistic and quantal level of the curved finsheaves of qausets. Moreover, our projective limit result—the* **`smooth`** *differential triad, Lorentzian principal sheaf and non-trivial connection on it which, as noted above, closely resembles the classical $\mathcal{C}^{\infty}$-diferential triad as well as the principal orthochronous Lorentz sheaf (bundle) and its associated curved locally Minkowskian vector sheaf (bundle) over the $\mathcal{C}^{\infty}$-smooth manifold M of general relativity—only illustrates the ability of our discrete algebraic (quantal) structures to yield at the (cor-*

[75] The word 'emulates' above pertains to the fact that our projective limit triad (as well as the principal sheaf and spin-Lorentzian connection relative to it) will be seen not to correspond precisely to the classical differential triad ($\mathbf{A}_X \equiv {}^{\mathbb{K}}\mathcal{C}^{\infty}_X, \partial, \mathbf{\Omega}^1$), but to one that in the context of the present ADG based paper may be regarded as a '*generalized smooth*' triad (write `smooth` for short). This `smooth` triad's structure sheaf will be symbolized by $\mathbf{A}_X \equiv {}^{\mathbb{K}}\mathfrak{C}^{\infty}_X$ in order to distinguish it from the ${}^{\mathbb{K}}\mathcal{C}^{\infty}_X$ employed in the classical case. On the other hand, we will be using the same symbols for the flat 0-th order nilpotent derivation $d^0 \equiv \partial$ as well as the $\mathbf{A}$-module of 1st order differential forms $\mathbf{\Omega}^1$ in the $\mathfrak{C}^{\infty}$-`smooth` and the usual $\mathcal{C}^{\infty}$-smooth triads.

[76] For, to recall Grauert and Remmert: "*The methods of sheaf theory are algebraic.*" [47]. The purely algebraic character of ADG has been repeatedly emphasized in [67, 68, 70, 77, 78, 74, 75, 73, 71].

> *respondence) limit of infinite localization or refinement of the qausets a structure that emulates well the kinematical structure of classical Lorentzian gravity* [92, 74, 93, 75]. At the same time, and perhaps more importantly, this indicates, in contrast to [8] where projective limits are employed in order to produce '*like from like*' (*ie*, induce a classical differential geometric structure from inverse systems of differential manifolds), what we have repeatedly stressed here, namely that, *to do differential geometry—the differential geometric machinery so to speak—is not inextricably tied to the $\mathcal{C}^\infty$-smooth manifold, so that we do not depend on the latter to provide us with the standard, and by no means unique, necessary or 'preferred', differential mechanism usually supplied by the algebra $\mathcal{C}^\infty(M)$ of smooth functions on the differential manifold M as in the classical case. Our differential geometric machinery, as we shall see in the sequel, comes straight from the (incidence) algebras inhabiting the stalks of vector, differential module and algebra sheaves like the generic locally free* **A***-modules $\mathcal{E}$ of ADG above, over a finitary topological base space(time) without mentioning at all any differential structure that this base space should a priori be equipped with, and certainly not the classical $\mathcal{C}^\infty$-manifold one. In other words, our differential geometric machinery does not come from assuming $\mathcal{C}^\infty_M$ as structure sheaf in our finitary, ADG-based constructions.*[77]

We would like to distill this to the following slogan that time and again we will encounter in the sequel:

> **Slogan 1.** Differentiability derives from (algebras in) the stalk (in point of fact, from the structure sheaf **A** of coefficients or generalized arithmetics), not from the base space.[78]

Then, the upshot of our approach to all the structures to be involved in the sequel is that

> in the spirit of ADG [67, 68, 70, 74, 75, 77, 78] and what has been presented so far here along those lines, *everything to be constructed below, whether kinematical or dynamical, is manifestly independent of a background $\mathcal{C}^\infty$-smooth spacetime manifold M, its 'structure group'* $\mathrm{Diff}(M)$ *and, as a result, of the usual differential geometry (ie, calculus) that such a base space supports. In a nutshell, our (differential) geometric constructions are genuinely background $\mathcal{C}^\infty$-manifold free.*

Interestingly enough, such a position recurs time and again, as a *leit motiv* so to speak, in the Ashtekar quantum gravity program [4, 5]. But let us now go on to more details.

[77] A similar point was made in footnotes 8 and 64, for example. We will return to discuss it in more detail in the concluding section.

[78] As we have said many times, the classical case corresponding to taking for base space X (a region of) the smooth manifold M and for $\mathbf{A}_X$ its structure sheaf $\mathcal{C}^\infty_X$—the sheaf of germs of sections of infinitely differentiable functions on X.

4.1 Principal Finsheaves and their Associated Finsheaves of Qausets

First, we give a short account of the evolution of our ideas leading to [74] and [75] which the present paper is supposed to continue as it takes a step further into the dynamical realm of qausets.[79]

4.1.1 A brief history of finitary spacetime and gravity

Our entire project of developing a finitary, causal and quantal picture of spacetime and gravity started with Sorkin's work on discrete approximations of continuous spacetime topology [114]. Briefly, Sorkin showed that when one substitutes the point-events of a bounded region X of a topological (*ie*, $\mathcal{C}^0$) spacetime manifold M by 'coarse' regions (*ie*, open sets) U about them belonging to a locally finite open cover $\mathcal{U}_i$ of X, one can effectively replace the latter by locally finite partially ordered sets (posets) P_i which are T_0-topological spaces in their own right and, effectively, topologically equivalent to X. Then, these posets were seen to constitute inverse systems $\overleftarrow{\mathcal{P}} = (P_i, \succeq)$ of finitary topological spaces, with the relation $P_j \succeq P_i$ being interpreted as 'the act of topological refinement or resolution of P_i to P_j'.[80] Sorkin was also able to show, under reasonable assumptions about X,[81] that the P_is are indeed legitimate substitutes of it in that at the inverse or projective limit of infinite refinement, resolution or localization of the $\mathcal{U}_i$s and their associated P_is, one recovers the $\mathcal{C}^0$-region X (up to homeomorphism). Formally one writes

$$\varprojlim \overleftarrow{\mathcal{P}} \equiv \lim_{\infty \leftarrow i} P_i \equiv P_\infty \overset{\text{homeo.}}{\simeq} X \tag{105}$$

Subsequently, by exploring ideas related to Gel'fand duality,[82] which had already been anticipated in [129], Raptis and Zapatrin showed how to associate a finite dimensional, associative and noncommutative *incidence Rota* $\mathbb{K}$*-algebra* Ω_i with every P_i in $\overleftarrow{\mathcal{P}}$, and how these algebras can be interpreted as discrete and quantum topological spaces bearing a non-standard topology, called the *Rota topology*, on their primitive spectra[83] [92]. They also showed, in a way reminiscent of the Alexandrov-Čech construction of nerves associated with locally finite open covers of manifolds, how *the P_is may be also viewed as simplicial complexes*[84] as well as, again by exploring

[79] For a more detailed and thorough description of the conceptual history of our work, as well as of its relation with category and topos theory, the reader is referred to the recent work [90]. A topos-theoretic treatment of finitary, causal and quantal Lorentzian gravity is currently under way [91].

[80] Meaning essentially that the open covering $\mathcal{U}_i$ of X from which P_i derives is a subcover of (*ie*, coarser than) $\mathcal{U}_j$. Roughly, the latter contains more and 'smaller' open sets about X's points than the former. In this sense, acts of 'refinement', 'resolution', or 'localization' are all synonymous notions. That is, one refines the coarse open sets about X's point-events and in the process she localizes them (*ie*, she effectively determines their locus) at higher resolution or 'accuracy'. As befits this picture, Sorkin explicitly assumes that "*the points of X are the carriers of its topology*" [114].

[81] For instance, X was assumed to be *relatively compact* (open and bounded) and (at least) T_1.

[82] We will comment further on Gel'fand duality in the next section.

[83] That is, the sets of the incidence algebras' primitive ideals which, in turn, are kernels of irreducible representations of the Ω_is.

[84] See also [130, 128] about this.

a variant of Gel'fand duality, how there is a contravariant functor between the category $\mathfrak{P}$ of finitary substitutes P_i and poset morphisms[85] between them, and the category $\mathfrak{Z}$ of the incidence algebras Ω_i associated with the P_is and injective algebra homomorphisms between them. Below, we would like to highlight three issues from the investigations in [92]:

1. Since the Ω_is are objects dual to the P_is which, in turn, are discrete homological objects (*ie*, finitary simplicial complexes) as mentioned above, they (*ie*, the incidence algebras) can be viewed as *discrete differential manifolds* [26, 28, 27, 128]. Indeed, they were seen to be reticular spaces

$$\Omega_i = \bigoplus_{p\in\mathbb{Z}_+} \Omega_i^p = \overbrace{\Omega_i^0}^{\mathbb{A}_i} \oplus \overbrace{\Omega_i^1 \oplus \Omega_i^2 \oplus \ldots}^{\mathbb{D}_i} \equiv \mathbb{A}_i \oplus \mathbb{D}_i \tag{106}$$

   of $\mathbb{Z}_+$-graded $\mathbb{A}_i$-bimodules $\mathbb{D}_i$ of (exterior) differential forms Ω_i^p $(p \geq 1)$[86] related within each Ω_i by nilpotent Cartan-Kähler-like (exterior) differential operators $d_i^p: \ \Omega_i^p \longrightarrow \Omega_i^{p+1}$.

2. Since now the Ω_is are seen to be structures encoding not only topological, but also differential geometric information, it was intuited that an inverse—or more accurately, since the incidence algebras are objects Gel'fand-dual to Sorkin's topological posets—a direct system $\overrightarrow{\mathfrak{R}} = \{\Omega_i\}$ of the Ω_is should yield, now at the *direct* or *inductive limit* of infinite refinement of the $\mathcal{U}_i$s as in (105), an algebra Ω_∞ whose commutative subalgebra part $\mathbb{A}_\infty$ corresponds to ${}^{(\mathbb{K})}\mathcal{C}^\infty(X)$—the algebra of ($\mathbb{K} = \mathbb{R}, \mathbb{C}$-valued) smooth coordinates of the point-events of X, while Ω_∞^p in $\mathbb{D}_\infty$ to the ${}^{(\mathbb{K})}\mathcal{C}^\infty(X)$-bimodules of smooth differential p-forms cotangent at each and every point-event of X which, in turn, can now be regarded as being a smooth region of a $\mathcal{C}^\infty$-manifold M.[87] We will return to discuss further this limit in subsection 4.2.

[85] Monotone maps continuous in the topology of the P_is.

[86] In (106), $\mathbb{A}_i \equiv \Omega_i^0$ is a commutative subalgebra of Ω_i called *the algebra of coordinate functions in* Ω_i, while $\mathbb{D}_i \equiv \bigoplus_i^{p\geq 1} \Omega_i^p$ a linear subspace of Ω_i called *the module of differentials over* $\mathbb{A}_i$. The elements of each linear subspace Ω_i^p of Ω_i in $\mathbb{D}_i$ were seen to be discrete analogues of (exterior) differential p-forms. We also note that in the sequel we will use the same boldface symbol '$\mathbb{A}_i$' and '$\mathbb{D}_i$' to denote the algebra of reticular coordinates and the module of discrete exterior differentials over it as well as the finsheaves thereof.

[87] In retrospect, and as we shall see in the sequel from an ADG-theoretic perspective, that initial anticipation in [92, 93]—that is, that at the inductive limit of infinite localization of the Ω_is one should recover the classical smooth structure of a $\mathcal{C}^\infty$-manifold—was wrong, or better, slightly misled by the classical $\mathcal{C}^\infty$-theory. In fact, as noted earlier, based on ADG results about inverse and direct limits of differential triads, we will argue subsequently that at the continuum limit one recovers a `smooth` algebra structure ${}^{\mathbb{K}}\mathcal{C}^\infty(X)$ and ${}^{\mathbb{K}}\mathcal{C}^\infty(X)$-bimodules Ω_∞^p of `smooth` p-forms over it, and that both of which may be regarded as 'generalized', albeit close, relatives of the corresponding classical $\mathcal{C}^\infty$-ones. Thus, rather than directly anticipate that one should obtain the local smooth structure of a $\mathcal{C}^\infty$-manifold at the inductive limit of infinite refinement (of the incidence algebras), perhaps it is more correct at this point just to emphasize that passing from the poset to the incidence algebraic regime one catches a glimpse not only of the topological, but also of the differential structure of discretized spacetime. This essentially shows that *the differential operator*—the heart and soul of differential geometry—*comes straight from the algebraic structure*. Equivalently, *incidence algebras provide us with a (reticular) differential geometric mechanism*, something that the 'purely topological' finitary posets were unable to supply since they are merely

3. The aforesaid continuum limit was physically interpreted as Bohr's correspondence principle, in the following sense: the local (differential) structure of classical $\mathcal{C}^\infty$-smooth spacetime should emerge at the physically 'ideal' (or operationally 'non-pragmatic') limit of infinite localization of the alocal, discrete and quantal algebraic substrata Ω_i.[88]

In the sequel, following Sorkin's dramatic change of physical interpretation of the locally finite posets P_i in [114] from finitary topological spaces to *causal sets* (causets) $\vec{P}_i$ [18],[89] the corresponding reticular and quantal topological spaces Ω_i where similarly interpreted as *quantum causal sets* (qausets) $\vec{\Omega}_i$ [86].[90] Qausets, like their causet counterparts, were regarded as locally finite, causal and quantal substrata underlying the classical Lorentzian spacetime manifold of macroscopic gravity.[91] On the other hand, it was realized rather early, almost ever since their inception in [18, 116], that causets are sound models of the *kinematical structure* of (Lorentzian) spacetime in the quantum deep, so that in order to address genuinely dynamical issues *vis-à-vis quantum gravity*, causet theory should also suggest a dynamics for causets. Thus,

how can one vary a locally finite poset?

has become the main question in the quest for a dynamics for causets[92] [90].

It was roughly at that point, when the need to develop a dynamics for causets arose, that ADG entered the picture. In a nutshell, we intuited that a possible, rather general answer to the question above, is

by sheaf-theoretic means!

in the sense that the fundamentally algebraic methods of sheaf theory, as employed by ADG, could be somehow used to model a realm of dynamically varying causets or, preferably, due to a quantum theoresis of (local) causality and gravity that we had in mind, of their qauset descendants.

associative multiplication structures (*ie*, arrow semigroups, or monoids, or even poset categories) and not linear structures (*ie*, one is not able to form differences of elements in them). This remark will be of crucial importance subsequently when we will apply ADG-theoretic ideas to these discrete differential algebras.

[88]For further remarks on this limiting procedure and its physical interpretation, the reader is referred to [92, 74, 93, 75, 131]. We will return to it in an ADG-theoretic context in the next subsection where, as noted above, we will show that one does not actually get the classical $\mathcal{C}^\infty$-smooth structure at the continuum limit, but a $\mathcal{C}^\infty$-`smooth` one akin to it. We will also argue that this (*ie*, that we do not get back the $\mathcal{C}^\infty$-smooth spacetime manifold at the projective/inductive limit of our finitary structures) is actually welcome when viewed from the ADG perspective of the present paper.

[89]For a thorough account of this semantic switch from posets as discrete topologies to posets as locally finite causal spaces, the reader is referred to [115].

[90]The reader should note that, in accordance with our convention in [86, 74, 75], from now on all our constructions referring to reticular *causal* structures like the $\vec{P}_i$s and their associated $\vec{\Omega}_i$s, will bear a right-pointing arrow over them just to remind us of their causal interpretation. (Such causal arrows should not be confused with the right-pointing arrows over inductive systems.)

[91]That causality, as a partial order, determines not only the topology and differential structure of the spacetime manifold as alluded to above, but also its conformal Lorentzian metric structure of (absolute) signature 2, has been repeatedly emphasized in [18, 113, 116, 117].

[92]Rafael Sorkin in private correspondence.

However, in order to apply the concrete sheaf-theoretic ideas and techniques of ADG to qausets, it was strongly felt that we should somehow marry first Sorkin's original finitary posets in [114] with sheaves proper. Thus, *finitary spacetime sheaves* (finsheaves) were defined as spaces S_i of (algebras of) continuous functions on Sorkin's T_0-posets P_i that were seen to be *locally homeomorphic* to each other [87].[93] The definition of finsheaves can be captured by the following commutative diagram which we borrow directly from [87]

$$\begin{array}{ccc} X & \xrightarrow{f_i} & P_i \\ {\scriptstyle \sigma\equiv\pi^{-1}}\downarrow & & \downarrow{\scriptstyle \pi_i^{-1}\equiv\sigma_i} \\ \mathcal{C}^0_X & \xrightarrow[\hat{f}_i]{} & S_{P_i} \end{array} \tag{107}$$

where $\mathcal{C}^0_X$ is the usual sheaf of germs of continuous functions on X, while f_i and $\hat{f}_i$ are continuous surjections from the topological spaces X and $\mathcal{C}^0_X$ to the finitary topological spaces P_i and S_i, respectively.

Now, the diagram (107) above prompts us to mention that the complete analogy between Sorkin's finitary topological posets P_i and finsheaves S_i rests on the result that an inverse system $\overleftarrow{\mathcal{S}}=(S_i,\hat{\succeq})$ of the latter was seen in [87] to possess a projective limit sheaf $S_\infty\equiv S_{P_\infty}$[94] that is homeomorphic to $\mathcal{C}^0_X$—the sheaf of germs of sections of continuous functions on the topological spacetime manifold X. That is to say, similarly to (105), one formally writes,

$$\varprojlim \overleftarrow{\mathcal{S}} \equiv \lim_{\infty\leftarrow i} S_i \equiv S_\infty \overset{\text{homeo.}}{\simeq} \mathcal{C}^0_X \tag{108}$$

One could cast the result above as a limit of commutative diagrams like the one in (107) which

[93]That is, one formally writes $P_i \ni U \underset{\pi_i}{\overset{\sigma_i}{\rightleftarrows}} S_i(U)$, where π_i is the continuous projection map from the sheaf space S_i to the base topological poset P_i, σ_i its inverse (continuous local section) map and U an open subset of P_i. In other words, for every open U in P_i: $\pi_i\circ\sigma_i(U)=U \Leftrightarrow [\forall U\in P_i:\ \sigma_i=\pi_i^{-1}]$ (*ie*, σ_i is a local homeomorphism having π_i for inverse) [87, 67]. Here we symbolize these finsheaves by $S_i\equiv S_{P_i}$.

[94]From (105), $P_\infty \overset{\text{homeo.}}{\simeq} X$.

defines finsheaves, as follows

$$
\begin{array}{ccc}
P_i & \underset{\sigma_i}{\overset{\pi_i^{-1}}{\longrightarrow}} & S_i \\
f_{ij}\big\downarrow \succeq_{ij} & & \hat{\succeq}_{ij}\big\downarrow \hat{f}_{ij}\\
P_j & \underset{\sigma_j}{\overset{\pi_j^{-1}}{\longrightarrow}} & S_j \\
\vdots & & \vdots \\
f_{j\infty}\circ f_{ij}=:f_{i\infty}\big\downarrow \succeq_{i\infty} & & \hat{\succeq}_{i\infty}\big\downarrow \hat{f}_{i\infty}:=\hat{f}_{j\infty}\circ\hat{f}_{ij}\\
\lim\limits_{\infty\leftarrow i} P_i \equiv P_\infty \overset{\text{homeo.}}{\simeq} X & \underset{\sigma}{\overset{\pi^{-1}}{\longrightarrow}} & \mathcal{C}^0_X \overset{\text{homeo.}}{\simeq} S_\infty \equiv \lim\limits_{\infty\leftarrow i} S_i
\end{array}
\tag{109}
$$

with f_{ij} and $\hat{f}_{ij}$ continuous injections—the 'refinement' or 'localization arrows'—between the P_is in $\overleftarrow{\mathcal{P}}$ and the S_is in $\overleftarrow{\mathcal{S}}$, respectively.[95]

Having finsheaves in hand, our next goal was to materialize ADG-theoretically our general answer to Sorkin's question mentioned above. The basic idea was the following:

> Since sheaves of (algebraic) objects of any kind may be regarded as universes of variable objects [67, 64], by (sheaf-theoretically) localizing or 'gauging' the incidence Rota algebras modelling qausets over the finitary topological posets P_i or their locally finite causet descendants $\vec{P}_i$,[96] the resulting finsheaves would stand for worlds of variable qausets—ones varying dynamically under the influence of a locally finite, causal and quantal version of gravity *in vacuo* which, in turn, could be concisely encoded in non-flat connections on those finsheaves [74]. Moreover, and this cannot be overemphasized here, by using the rather universal sheaf-theoretic constructions

[95] These arrows capture precisely the partial order (or net) refinement relations $\succeq$ and $\hat{\succeq}$ between the finitary posets in $\overleftarrow{\mathcal{P}}$ and their corresponding finsheaves in $\overleftarrow{\mathcal{S}}$ respectively, as (109) depicts (*eg*, we formally write: $P_i \xrightarrow{f_{ij}} P_j \equiv P_j \succeq_{ij} P_i$). Also from (109), one notices what we said earlier in connection with (105) and (108), namely, that X and $\mathcal{C}^0_X$ are obtained at the categorical limit of infinite (topological) refinement or localization ($\succeq_{i\infty}$ and $\hat{\succeq}_{i\infty}$) of the P_is and the S_is, respectively.

[96] For instance, one could regard $\vec{P}_i$ as a topological space proper by assigning a 'causal topology' to it, as for example, by basing such a topology on 'open' sets of the following kind: $I^-(x) := \{y \in \vec{P}_i :\ y \rightarrow x\}$ $(\forall x \in \vec{P}_i)$ ('lower' or 'past-set topology'), or dually on: $I^+(x) := \{y \in \vec{P}_i :\ x \rightarrow y\}$ ('upper' or 'future-set topology'), or even on a combination of both—*ie*, on 'open' causal intervals of the following sort: $A(x,y) := I^+(x) \cap I^+(y)$ (the so-called Alexandroff topology). It is one of the basic assumptions about the causets of Sorkin *et al.* that the cardinality of the Alexandroff sets $A(x,y)$ is finite—the so-called *local finiteness* property of causets [18]. As basic open sets generating the three topologies above, one could take the so-called *covering past*, *covering future* and *null Alexandroff* 'open' sets, respectively. These are $I_c^-(x) := \{y \in \vec{P}_i :\ (y \rightarrow x) \wedge (\nexists z \in \vec{P}_i :\ y \rightarrow z \rightarrow x)\}$, $I_c^+(x) := \{y \in \vec{P}_i :\ (x \rightarrow y) \wedge (\nexists z \in \vec{P}_i :\ x \rightarrow z \rightarrow y)\}$ and $A_0(x,y) = \emptyset \Leftrightarrow (x \rightarrow y) \wedge (\nexists z \in \vec{P}_i :\ x \rightarrow z \rightarrow y)$, respectively. (Note: the *immediate arrows* in the Hasse diagram of any poset P appearing in the definition of I_c^-, I_c^+ and $A_n(x,y)$ are called *covering relations* or *links* and they correspond to the *transitive reduction* of the partial order based at each vertex in the directed and transitive graph of P. In turn, the three topologies mentioned above can be obtained by taking the *transitive closure* of these links [86, 75].)

of ADG, we could carry virtually all the usual $\mathcal{C}^\infty$-differential geometric machinery on which the mathematical formulation of general relativity rests, to the locally finite setting of finsheaves of qausets [75]—the principal differential geometric objects being, of course, the aforesaid connections on the relevant finsheaves, which implement the dynamics of qausets.

Thus, as a first step in this development, we set out to define *(curved) principal finsheaves* $\vec{\mathcal{P}}_i^{\uparrow} := \overrightarrow{\mathcal{A}ut}_{\vec{\mathbf{A}}_i}\vec{\mathbf{\Omega}}_{\vec{P}_i} \equiv \overrightarrow{\mathcal{A}ut}_i\vec{\mathbf{\Omega}}_i$ *of qausets, and their associated finsheaves* $\vec{\mathbf{\Omega}}_{\vec{P}_i} \equiv \vec{\mathbf{\Omega}}_i$*, over a causet* $\vec{P}_i$.[97] By establishing finitary versions of the classical general relativistic principles of equivalence and locality, we realized that the (local) structure (gauge) symmetries of $\vec{\mathbf{\Omega}}_i$ are finitary correspondents of the orthochronous Lorentz Lie group (*ie*, locally in $\vec{P}_i$ one writes formally: $\overrightarrow{\mathcal{A}ut}_{\vec{\mathbf{A}}_i}\vec{\mathbf{\Omega}}_{\vec{P}_i}(U) = SO(1,3)_i^{\uparrow})$,[98] and that they could thus be organized into the aforesaid $\mathcal{G}_i$-finsheaves $\vec{\mathcal{P}}_i^{\uparrow}$. Then, by definition, the $\vec{\mathbf{\Omega}}_i$s are the associated finsheaves of the principal $\vec{\mathcal{P}}_i^{\uparrow}$s.

From the start we also realized that the localization or 'gauging' of qausets in the $\vec{\mathcal{P}}_i^{\uparrow}$s and their associated $\vec{\mathbf{\Omega}}_i$s meant that these finsheaves could be endowed with non-trivial (*ie*, non-flat) reticular spin-Lorentzian connections $\vec{\mathcal{D}}_i$ *à la* ADG. Indeed, in complete analogy to the general ADG case, after having defined reticular flat connections as the following $\mathbf{K}$-linear and section-wise Leibniz condition (2)-obeying finsheaf morphisms

$$\vec{d}_i^0 \equiv \vec{\partial}_i : \ \vec{\mathbf{\Omega}}_i^0 \equiv \vec{\mathbf{A}}_i \longrightarrow \vec{\mathbf{\Omega}}_i^1 \tag{110}$$

as in (1), as well as higher order extensions

$$\vec{d}_i^p : \ \vec{\mathbf{\Omega}}_i^p \longrightarrow \vec{\mathbf{\Omega}}_i^{p+1}, \ (\mathbb{N} \ni p \geq 1) \tag{111}$$

between the vector subsheaves $\vec{\mathbf{\Omega}}_i^p$ of $\vec{\mathbf{\Omega}}_i$, we defined in [74] non-flat connections $\vec{\mathcal{D}}_i$ on the finsheaves $\vec{\mathbf{\Omega}}_i$ of finite dimensional differential $\vec{\mathbb{A}}_i$-bimodules $\vec{\Omega}_i$[99] again as the following $\mathbf{K}$-linear and section-wise Leibniz condition-obeying (4) finsheaf morphisms

$$\vec{\mathcal{D}}_i : \ \vec{\mathcal{E}}_i \equiv \vec{\mathbf{\Omega}}_i^* \longrightarrow \vec{\mathcal{E}}_i \otimes_{\vec{\mathbf{A}}_i} \vec{\mathbf{\Omega}}_i \equiv \vec{\mathbf{\Omega}}_i(\vec{\mathcal{E}}_i) \tag{112}$$

similarly to (3).[100] Moreover, in complete analogy to the local expression for the abstract $\mathcal{D}$s in

[97] In what follows we will be often tempted to use the same epithet, '*principal*', for both the $\vec{\mathcal{P}}_i^{\uparrow}$s and their associated $\vec{\mathbf{\Omega}}_i$s. We do hope that this slight abuse of language will not confuse the reader. As we will see in the sequel, this identification essentially rests on our assuming a general Kleinian stance towards (physical) geometry whereby 'states' (of a physical system) and the 'symmetry group of transformations of those states' are regarded as being equivalent.

[98] Where U is an open set in $\vec{P}_i$ regarded as a causal-topological space (see footnote 96 above).

[99] The reader should have gathered by now that in the stalks of the structure finsheaves $\vec{\mathbf{A}}_i$ dwell the (causal versions $\vec{\mathbb{A}}_i$ of the) abelian (sub)algebras $\mathbb{A}_i$ (of Ω_i) in (106), while in the fibers of $\vec{\mathbf{D}}_i$ the (causal versions $\vec{\mathbb{D}}_i$ of the) $\mathbb{A}_i$-modules $\mathbb{D}_i$ in (106).

[100] The reader should note in connection with (112) that the 'identification' $\vec{\mathcal{E}}_i \equiv \vec{\mathbf{\Omega}}_i^*$ tacitly assumes that there is a (Lorentzian) metric $\vec{\rho}_i$ on the vector sheaves $\vec{\mathcal{E}}_i$ effecting canonical isomorphisms $\tilde{\vec{\rho}}_i$ between them and their dual differential module (covector) finsheaves $\vec{\mathbf{\Omega}}_i$, as in (12). We will give more details about $\vec{\rho}_i$ and the implicit

(8), the finitary $\vec{\mathfrak{D}}_i$s were seen to split locally to

$$\vec{\mathfrak{D}}_i = \vec{\partial}_i + \vec{\mathcal{A}}_i,\ (\vec{\mathcal{A}}_i \in \vec{\mathbf{\Omega}}^1_i(U),\ U \text{ open in } \vec{P}_i) \tag{113}$$

and the reticular gauge potentials $\vec{\mathcal{A}}_i$ of the $\vec{\mathfrak{D}}_i$s above were readily seen to be $\overrightarrow{\mathcal{A}ut_i}$-valued local sections of $\vec{\mathbf{\Omega}}^1_i$ (*ie*, 'discrete' $so(1,3)^{\uparrow}_i \simeq sl(2,\mathbb{C})_i$-valued local 1-forms),[101] in analogy with both the classical and the abstract (ADG) theory.

At this point, we must stress a couple of things about these finitary spin-Lorentzian connections $\vec{\mathfrak{D}}_i$ *vis-à-vis* the general ADG theory presented in the previous two sections.

1. **About the base space.** As it was mentioned in [87], [74] and [75], in our finitary regime there are mild relaxations of the two basic conditions of *paracompactness* and *Hausdorffness* (T_2-ness) that ADG places on the base topological space X on which the vector sheaves $\mathcal{E}$ bearing connections $\mathcal{D}$ are soldered. As noted in footnote 81, the starting region X of the topological spacetime manifold M from which the $\vec{P}_i$s (and their associated $\vec{\Omega}_i$s) come from was assumed in [114] to be *relatively compact* and (at least) T_1. If one relaxes paracompactness to relative compactness, and T_2-ness to T_1-ness (and we are indeed able to do so without any loss of generality),[102] one is still able to carry out in the locally finite regime the entire spectrum of the ADG-theoretic constructions described in the last two sections.[103]

identification of the finitary vectors in $\vec{\mathcal{E}}_i$ with their corresponding forms in $\vec{\mathbf{\Omega}}_i$ shortly. For the time being, we note that we would like to call $\vec{\mathfrak{D}}_i$ '*the (f)initary, (c)ausal and (q)uantal (v)acuum dynamo*' (fcqv-dynamo) for a reason to be explained in the next section.

[101] Of course, since the $\vec{\mathbf{\Omega}}_i$s are curved, they do not admit global sections [67, 74]. In view of the name 'fcqv-dynamo' we have given to $\vec{\mathfrak{D}}_i$ in the previous footnote, its gauge potential part $\vec{\mathcal{A}}_i$ may be fittingly coined a '*fcqv-potential*'. The fcqv-potential, like its abstract analogue ω in (6)–(8), is an $n \times n$-matrix of sections of local reticular 1-forms (*ie*, $\vec{\mathcal{A}}_i \equiv (\vec{\mathcal{A}}^i_{pq}) \in M^i_n(\vec{\mathbf{\Omega}}^1_i(U))$, U open in $\vec{P}_i$). Also, since the local structure of the gauge group $\mathcal{G}_i$ of the $\vec{\mathbf{\Omega}}_i$s is the reticular orthochronous Lorentz Lie algebra $so(1,3)^{\uparrow}_i$, we will denote the vector finsheaves $\vec{\mathcal{E}}_i$ above as $\vec{\mathcal{E}}^{\uparrow}_i = (\vec{\mathcal{E}}_i, \vec{\rho}_i)$, in accord with our notation earlier for the (real) orthochronous Lorentzian vector sheaves $\mathcal{E}^{\uparrow} = (\mathcal{E}, \rho)$ of rank 4 in the context of ADG. (However, to avoid uncontrollable proliferation of symbols and eventual typographical congestion of indices, superscripts *etc*, we will not denote the dual spaces $\vec{\mathbf{\Omega}}_i$s of the $\vec{\mathcal{E}}^{\uparrow}_i$s by $\vec{\mathbf{\Omega}}^{\uparrow}_i$.) Moreover, notice that, as it was mentioned in [74], the 'finitarity index i' on $so(1,3)^{\uparrow}_i$ indicates that the Lie group manifold $SO(1,3)^{\uparrow}$ of (local) structure gauge symmetries of the qausets is also subjected to discretization as well. It is reasonable to assume that *finitary structures have finitary symmetries* or equivalently and perhaps more popularly, *discrete structures possess discrete symmetries*. This is in accord with our abiding to a Kleinian conception of (physical) geometry, as noted in footnote 97. On the other hand, we shall see in the next section that the finitarity index indicates only that our structures are 'discrete' and *not* that they are essentially dependent on the locally finite covering (gauge) $\mathcal{U}_i$ of X. In fact, we will see that (from the dynamical perspective) our constructions are *inherently gauge $\mathcal{U}_i$-independent* and for this reason '*alocal*' [92, 74, 93]. In other words, the (dynamical) role played by the base localization causet $\vec{P}_i$ and, *in extenso*, by the region X of the Lorentzian spacetime manifold that the latter discretizes relative to $\mathcal{U}_i$, is physically insignificant.

[102] In fact, as noted in both [92] and [93], at the finitary poset level one must actually insist on relaxing Hausdorffness, because a T_2-finitary substitute in [114] is automatically trivial as a topological space—that is, it carries the discrete topology, or equivalently, it is a completely disconnected set (no arrows between its point vertices).

[103] In fact, we could have directly started our finsheaf constructions straight from a paracompact and Hausdorff X without coming into conflict with Sorkin's results. For instance, already in [75] we applied the entire sheaf-

2. **About the stalk: Lorentzian metric and its orthochronous symmetries.** The stalks of the $\vec{\mathbf{\Omega}}_i$s are occupied by qausets $\vec{\Omega}_i$; in other words, they are the spaces where the (germs of the) continuous local sections of the $\vec{\mathbf{\Omega}}_i$s take values. These qausets, as it has beeen argued in [74], determine a metric $\vec{\rho}_i$ of Lorentzian signature. Thus, as it was emphasized in footnote 17 of 2.2, $\vec{\rho}_i$ is not carried by the base space $\vec{P}_i$, which is simply a topological space; rather, it concerns directly the (objects living in the stalks of the) relevant finsheaves *per se*. In fact, we may define this metric to be the following finsheaf morphism:

$$\vec{\rho}_i : \ \vec{\mathcal{E}}_i^{\uparrow} \oplus \vec{\mathcal{E}}_i^{\uparrow} \longrightarrow \vec{\mathbf{A}}_i \tag{114}$$

which, like its abstract version ρ in (11), is $\vec{\mathbf{A}}_i$-bilinear between the (differential) $\vec{\mathbf{A}}_i$-modules $\vec{\mathbf{\Omega}}_i$ concerned and (section-wise) symmetric.[104] It follows that the $\vec{\mathbf{A}}_i$-metric preserving (local) automorphism group finsheaf $\overrightarrow{\mathcal{A}ut}_{\vec{\mathbf{A}}_i}\vec{\mathcal{E}}_i^{\uparrow}|_{U\in\vec{P}_i} \equiv \overrightarrow{\mathcal{A}ut}_{\vec{\rho}_i}\vec{\mathcal{E}}_i|_{U\in\vec{P}_i}$ is the aforesaid principal $\vec{\mathcal{G}}$-finsheaf $\vec{\mathcal{P}}_i^{\uparrow}(U) \equiv \overrightarrow{\mathcal{A}ut}_{\vec{\rho}_i}\vec{\mathcal{E}}_i(U) \equiv \mathrm{SO}(1,3;\vec{\mathbf{A}}_i(U))_i^{\uparrow}$ of reticular orthochronous isometries of the (real) Lorentzian finsheaf $\vec{\mathcal{E}}_i^{\uparrow} = (\vec{\mathcal{E}}_i, \vec{\rho}_i)$ of rank 4.[105]

Also, in accordance with Sorkin *et al.*'s remark in [18] that a (locally finite) partial order determines not only the topological and the metric structure of the Lorentzian manifold of general relativity, but also its differential structure, we witness here that the aforementioned nilpotent Cartan-Kähler (exterior) differentials $\vec{d}_i^p$, which as we saw in (111) effect vector subsheaf morphisms $\vec{d}_i^p : \ \vec{\mathbf{\Omega}}_i^p \longrightarrow \vec{\mathbf{\Omega}}_i^{p+1}$ ($\mathbb{Z} \ni p \geq 0$), derive directly from the algebraic structure of the $\vec{\mathbf{\Omega}}_i$s—that is to say, again straight from the stalk of the finsheaves of qausets without any dependence on the base causet $\vec{P}_i$ which is simply a causal-topological space. We cannot overemphasize this either:

> *Differentiability in our finitary scheme, and according to ADG, does not depend on the base space (which is assumed to be simply a topological space); the differential mechanism comes staright from the stalk (ie, from the algebraic objects dwelling in it) and, a fortiori, certainly not from a classical, $\mathcal{C}^\infty$-smooth base spacetime manifold.*

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cohomological panoply of ADG to our finsheaves of qausets.

[104] In connection with footnote 100, we note that we tacitly assume that $\vec{\mathcal{E}}_i^{\uparrow} = (\vec{\mathcal{E}}_i, \vec{\rho}_i)$ in (114) is the dual to $\vec{\mathbf{\Omega}}_i$ (*ie*, $\vec{\mathbf{\Omega}}_i = \vec{\mathcal{E}}_i^{\uparrow *} = \mathcal{H}om_{\vec{\mathbf{A}}_i}(\vec{\mathcal{E}}_i^{\uparrow}, \vec{\mathbf{A}}_i)$). It is also implicitly assumed that $\vec{\rho}_i$ in (114) induces a canonical isomorphism between $\vec{\mathcal{E}}_i^{\uparrow}$ and its dual $\vec{\mathbf{\Omega}}_i$ analogous to (12). Thus, with a certain abuse of language, but hopefully without causing any confusion, we will assume that $\vec{\mathbf{\Omega}}_i \equiv \vec{\mathcal{E}}_i^{\uparrow}$ (*ie*, we identify via $\vec{\rho}_i$ finitary covectors and vectors) and use them interchangeably in what follows.

[105] Since, as noted in footnote 15, specific dimensionality arguments do not interest us here as long as the algebras involved in the stalks of our finsheaves are (and they are indeed) finite dimensional, the reader may feel free to choose an arbitrary, finite rank n for our finsheaves. Then, the reticular Lorentzian $\vec{\mathbf{A}}_i$-metric $\vec{\rho}_i$ involved will be of absolute signature $n-2$ (*ie*, $\vec{\rho}_i = \mathrm{diag}(-1, \overbrace{+1,+1,\ldots+1}^{n-1})$) and its local invariance (structure) group $\mathrm{SO}(1,n-1;\vec{\mathbf{A}}_i(U))^{\uparrow}$ (U open in $\vec{P}_i$, as usual).

3. **About the physical interpretation.** We would like to comment a bit on the physical interpretation of our principal finsheaves of qausets and the reticular spin-Lorentzian connections on them. First we must note that Sorkin *et al.*, after the significant change in physical interpretation of the locally finite posets involved from topological in [114] to causal in [18, 113, 115, 116, 117] alluded to above, insisted that, while the topological posets can be interpreted as coarse approximations to the continuous spacetime manifold of macroscopic physics, the causets should be regarded as being truly fundamental structures in the sense that the macroscopic Lorentzian manifold of general relativity is an approximation to the deep locally finite causal order, not the other way around.

   Our scheme strikes a certain balance between these two poles. For instance, while we assume a base causet on which we solder our incidence algebras modelling qausets, that causet is also assumed to carry a certain topology—the 'causal topology'[106]—so that it can serve as the background topological space on which to solder our algebraic structures, which in turn enables us to apply ADG to them thus unveil potent differential geometric traits of the qausets in the stalks, as described above. This causal topology however, in contradistinction to Sorkin's T_0-topological posets which model "*thickened space-like hypersurfaces*" in continuous spacetime [114], is regarded as a theory of 'thickened' causal regions in space*time* [86, 74, 93].[107] Furthermore, as it has been emphasized in [74], while the non-flat reticular spin-Lorentzian connections $\vec{\mathcal{D}}_i$ on the corresponding $\vec{\mathbf{\Omega}}_i$s can be interpreted as the fundamental operators encoding the *curving of quantum causality* thus setting the kinematics for a dynamically variable quantum causality, an inverse system $\overleftarrow{\mathcal{G}} := \{(\vec{\mathcal{P}}_i^{\uparrow}, \vec{\mathcal{D}}_i)\}$ was intuited to 'converge' at the operationally ideal (*ie*, non-pragmatic and 'classical' in Bohr's 'correspondence principle' sense [92]) limit of infinite refinement or localization of both the base causets and the associated qauset fibers over them to the classical principal fiber bundle $(\mathcal{P}^{\uparrow}, \mathcal{D})$ of continuous local orthochronous Lorentz symmetries $so(1,3)^{\uparrow}$ of the $\mathcal{C}^{\infty}$-smooth spacetime manifold M of general relativity and the $sl(2,\mathbb{C})$-valued spin-Lorentzian gravitational connection $\mathcal{D}$ on it.[108] Since $(\mathcal{P}^{\uparrow}, \mathcal{D})$ is the gauge-theoretic version of the kinematical structure of general relativity—the dynamical theory of the classical field of local causality $g_{\mu\nu}$,[109] each individual member $(\vec{\mathcal{P}}_i^{\uparrow}, \vec{\mathcal{D}}_i)$ of the inverse system $\overleftarrow{\mathcal{G}}$ was interpreted as the kinematics of a locally finite, causal and quantal version of (vacuum) Einstein Lorentzian

[106] See footnote 96.

[107] For more on this, see subsection 4.3 below.

[108] For more technical details about the projective limit of $\overleftarrow{\mathcal{G}}$, the reader must wait until the following subsection. At this point it must be stressed up-front, in connection with footnote 75, that what we actually get at the projective limit of $\overleftarrow{\mathcal{G}}$ is a $\mathcal{C}^{\infty}$-`smooth` principal bundle (and its spin-Lorentzian connection) over the region X of a 'generalized differential manifold' (*ie*, $\mathcal{C}^{\infty}$-`smooth`) M.

[109] For recall that the spacetime metric $g_{\mu\nu}(x)$, for every $x \in M$, delimits a Minkowski lightcone based at x (by the equivalence principle, the curved gravitational spacetime manifold of general relativity is, locally, Minkowski space, *ie*, flat, and in this sense general relativity may be viewed as special relativity being localized or 'gauged'). Thus, the Einstein equations of general relativity, which describe the dynamics of $g_{\mu\nu}$ (which, in turn, can be interpreted as the field of the ten gravitational potentials), effectively describe the dynamical changes of (the field of) local causality. All this was analyzed in detail in [74].

gravity.[110] *In toto*, we have amalgamated aspects from the interpretation of both the finitary substitutes and the causets, as follows:[111]

> *'Coarse causal regions' are truly fundamental, operationally sound and physically pragmatic, while the classical pointed $\mathcal{C}^\infty$-smooth spacetime manifold ideal.*[112] *Curved finsheaves of qausets ($\vec{\mathcal{P}}_i^{\uparrow} \equiv \vec{\mathcal{E}}_i^{\uparrow}, \vec{\mathcal{D}}_i$) model the kinematics of dynamical (local) quantum causality in vacuo as the latter is encoded in the $fcqv$-dynamo $\vec{\mathcal{D}}_i$. A generalized (ie, $\mathcal{C}^\infty$-**smooth**) version of the classical kinematical structure of general relativity, $(\mathcal{P}^{\uparrow}, \mathcal{D})$, over the differential spacetime manifold M, arises at the ideal and classical (Bohr's correspondence) limit of infinite localization of the qausets—in point of fact, of $\overleftarrow{\mathcal{G}}$.*[113]

4. **About 'reticular' differential geometry.** The basic moral of our application of ADG to the finitary regime as originally seen in [74] as well as here, but most evidently in [75], is that the fundamental differential mechanism which is inherent in the differential geometry that we all are familiar with[114] is independent of $\mathcal{C}^\infty$-smoothness so that it can be applied in full to our inherently reticular models, or equally surprisingly, to spaces that appear to be ultra-singular and incurably pathological or problematic when viewed from the differential manifold's viewpoint [77, 78, 97]. In our case, what is startling indeed is that none of the usual 'discrete differential mathematics' (*eg*, difference calculus, finite elements or other related Regge calculus-type of methods) is needed in order to address issues of differentiability and to develop a full-fledged differential geometry in a (locally) finite setting. For instance, there appears to be no need for defining up-front '*discrete differential manifolds*' and for developing *a priori* and, admittedly, in a physically rather *ad hoc* manner a '*discrete differential geometry*' on them[115] in order to investigate differential geometric properties of 'finitary' (ordered) spaces.[116] For they too can be cast under the wider axiomatic, algebraico-sheaf-theoretic prism of ADG as a particular application of the general theory. All in all, it is quite surprising indeed that the basic objects of the usual differential geometry like 'tangent' vectors (derivations), their dual forms, exterior

[110] As we shall see in the next section, the actual kinematical configuration space for the locally finite, causal and quantal vacuum Einstein gravity is the moduli space $\vec{\mathcal{A}}_i$ of finitary spin-Lorentzian connections $\vec{\mathcal{D}}_i$. As we shall see, projective limit arguments also apply to an inverse system of such reticular moduli spaces.

[111] Further distillation and elaboration on these ideas in subsection 4.3.

[112] More remarks on 'coarse causal regions' will be made in subsection 4.3.

[113] This is a concise *résumé* of a series of papers [92, 86, 74, 93, 75, 90]. Of course, 'infinite localization' requires 'infinite microscopic power' (*ie*, energy of determination or 'measurement' of locution) which is certainly an ideal (*ie*, operationally non-pragmatic and physically unattainable) requirement. This seems to be in accord with the pragmatic cut-offs of quantum field theory and the fundamental length L_P (the Planck length) that the 'true' quantum gravity is expected to posit (and below which it is expected to be valid!), for it is fairly accepted now that one cannot determine the locus of a quantum particle with uncertainty (error) less than $L_P \approx 10^{-35}m$ without creating a black hole. This seems to be the *raison d'être* of all the so-called 'discrete' approaches to quantum spacetime and gravity [74].

[114] Albeit, just from the classical (*ie*, $\mathcal{C}^\infty$-smooth) perspective.

[115] Like for example the perspective adopted in [26, 28, 10, 27].

[116] Like graphs (directed, like our posets here, or undirected), or even finite structureless sets.

derivatives, Laplacians, volume forms *etc*, carry through to the locally finite scene and none of their discrete (difference calculus') analogues is needed, but this precisely proves the point:

One feels, perhaps 'instinctively' due to one's long time familiarity with and the numerous 'habitual' (but quite successful!) applications of the usual smooth calculus where the differential mechanism comes from the supporting space (*ie*, it is provided by the algebra $\mathcal{C}^\infty(M)$ of infinitely differentiable functions on the differential manifold M), that in the 'discrete' case too some novel kind of 'discrete differential geometry' must come from a 'discrete differential manifold'-type of base space—as if *the differential calculus follows from, or at least that it must be tailor-cut to suit, space*. In other words, in our basic working philosophy we have been misled by the habitual applications and the numerous successes of the smooth continuum into thinking that differentiability comes from, or that it is somehow vitally dependent on, the supporting space. By the present application of ADG to our reticular models we have witnessed how, quite on the contrary, *differentiability comes from the stalk*—*ie*, from algebras dwelling in the fibers of the relevant finsheaves—and it has nothing to do with the ambient space, which only serves as an auxiliary, and in no way contributing to the said differential mechanism, topological space for the sheaf-theoretic localization of those algebraic objects. The usual differential geometric concepts, objects and mechanism that relates the latter still apply in our reticular environment and, perhaps more importantly, in spite of it.

4.2 Projective Limits of Inverse Systems of Principal Lorentzian Finsheaves

Continuous limits of finitary simplicial complexes and their associated incidence algebras, regarded as discrete and quantal topological spaces [92, 93], have been studied recently in [130, 131]. In this subsection, always based on ADG, we present the projective limit of the inverse system $\overleftarrow{\mathcal{G}} = \{(\vec{\mathcal{P}}_i^\uparrow, \vec{\mathfrak{D}}_i)\}$ of principal Lorentzian finsheaves of qausets $\vec{\mathcal{P}}_i^\uparrow$ equipped with reticular spin-Lorentzian connections $\vec{\mathfrak{D}}_i$ which was supposed in [74] to yield the classical kinematical structure of general relativity in its gauge-theoretic guise—that is, the principal orthochronous spin-Lorentzian bundle over the (region X of the) $\mathcal{C}^\infty$-smooth spacetime manifold M of general relativity locally supporting an $sl(2,\mathbb{C})$-valued (self-dual) smooth connection (*ie*, gauge potential) 1-form $\mathcal{A}^{(+)}$. We center our study on certain results from a recent categorical account of projective and inductive limits in the category $\mathfrak{DT}$ of Mallios' differential triads in [81, 82], as well as on results from a treatment of projective systems of principal sheaves (and their associated vector sheaves) endowed with Mallios' **A**-connections in [124, 125, 126]. Then, we compare this inverse limit result, at least at a conceptual level and in a way that emphasizes the calculus-free methods and philosophy of ADG, with the projective limit of a projective family $\overleftarrow{\mathcal{M}}$ of compact Hausdorff differential manifolds employed in [8] in order to endow the moduli space $\mathcal{A}/\mathcal{G}$ of gauge-equivalent non-abelian Y-M and gravitational connections with a differential geometric

structure. In fact, we will maintain that an inverse system $\overleftarrow{\mathcal{M}}$ of our finitary moduli spaces should yield at the projective limit of infinite localization a generalized version (*ie*, a $\mathcal{C}^\infty$-`smooth` one) of the classical moduli space $\mathcal{A}_\infty^{(+)}$ of gauge-equivalent (self-dual) $\mathcal{C}^\infty$-connections on the region X of the smooth spacetime manifold M.

The concept-pillar on which ADG stands is that of a *differential triad* $\mathfrak{T} = (\mathbf{A}, \partial, \mathbf{\Omega})$ associated with a $\mathbf{K} = \mathbf{R}, \mathbf{C}$-algebraized space $(X, \mathbf{A})$. In ADG, *differential triads specialize to abstract differential spaces*, while the $\mathbf{A}$s in them stand for (structure sheaves of) *abstract differential algebras of generalized smooth or differentiable coordinate functions*, and they were originally born essentially out of realizing that

> *the classical differential geometry of a manifold X is deduced from its structure sheaf $\mathcal{C}_X^\infty$, the latter being for the case at issue the result of the very topological properties*[117] *of the underlying 'smooth' manifold X.*

Thus, in effect, the first author originally, and actually quite independently of any previous relevant work, intuited, built and subsequently capitalized on the fact that the algebra sheaf $\mathbf{A}$ of generalized arithmetics (or abstract coordinates) is precisely the structure that provides one with all the basic differential operators and associated 'intrinsic differential mechanism' one needs to actually do differential geometry—the classical, $\mathcal{C}^\infty$-smooth theory being obtained precisely when one chooses $\mathcal{C}_M^\infty$ as one's structure sheaf of coordinates.[118] Thus, the objects dwelling in the stalks of $\mathbf{A}$ may be perceived as *algebras of generalized (or abstract) 'infinitely differentiable' (or 'smooth') functions*, with the differential geometric character of the base localization space X left completely undetermined—in fact, it is regarded as being totally irrelevant to ADG.[119]

In [81], the differential triads of ADG were seen to constitute a category $\mathfrak{DT}$—*the category of differential triads*. Objects in $\mathfrak{DT}$ are differential triads and morphisms between them represent abstract differentiable maps. In $\mathfrak{DT}$ one is also able to form finite products and, unlike the category of smooth manifolds where an arbitrary subset of a (smooth) manifold is not a (smooth) manifold, one can show that every object $\mathfrak{T}$ in $\mathfrak{DT}$ has canonical subobjects [81]. More importantly however, in [82] it was shown that $\mathfrak{DT}$ is complete with respect to taking projective and inductive limits of projective and inductive systems of triads, respectively.[120] This is a characteristic difference between $\mathfrak{DT}$ and the category of manifolds where the projective limit of an inverse

[117] Poincaré lemma [77, 75].

[118] Yet, we can still note herewith that the first author arrived at the notion of a *differential triad* as a particularization to the basic differentials of the classical theory of the amply ascertained throughout the same theory instrumental role played by the notion of an $\mathbf{A}(\equiv \mathcal{C}_X^\infty)$-connection (*ie*, covariant differentiation).

[119] Of course, as also noted earlier in footnote 64, in the classical case (*ie*, when one identifies $\mathbf{A}_M \equiv \mathcal{C}_M^\infty$) there is a confusion of the sort '*who came first the chick or the egg?*', since one tends to identify the underlying space(time) (*ie*, the $\mathcal{C}^\infty$-smooth manifold M) with its structure sheaf $\mathcal{C}_M^\infty$ of smooth functions and, more often than not, one is (mis)led into thinking that *differentiability—the intrinsic mechanism of differential geometry* so to speak—comes (uniquely!) from the underlying smooth manifold. This is precisely what ADG highlighted: *differentiability comes in fact from the structure sheaf, so that if one chooses 'suitable' or 'appropriate' (to the problem one chooses to address) algebras of 'generalized smooth' functions other than $\mathcal{C}^\infty(M)$, one is still able to do differential geometry (albeit, of a generalized or abstract sort) in spite of the classical, $\mathcal{C}^\infty$-smooth base manifold.*

[120] In fact, Papatriantafillou showed that projective/inductive systems of differential triads having either a com-

system of manifolds is not, in general, a manifold.[121] Moreover, Vassiliou, by applying ADG-theoretic ideas to principal sheaves (whose associated sheaves are precisely the vector sheaves of ADG) [124, 125, 126], has shown that when the flat differentials $\dot{\partial}$ of the triads in the aforesaid projective/inductive systems of Papatriantafillou are promoted (*ie*, 'gauged' or 'curved') to **A**-connections $\dot{\mathcal{D}}$ on principal sheaves, the corresponding projective/inductive systems $(\mathcal{P}_i, \dot{\mathcal{D}}_i)$[122] have principal sheaves endowed with non-flat connections as inverse/direct limits.

Thus, in our locally finite case, the triplet $\vec{\mathfrak{T}}_i = (\vec{\mathbf{A}}_{\vec{P}_i} \equiv \vec{\mathbf{A}}_i, \vec{\mathbf{D}}_{\vec{P}_i} \equiv \vec{\mathbf{D}}_i, \vec{d}_i^p)$ is an ADG-theoretic differential triad of a (f)initary, (c)ausal and (q)uantal kind. In other words, the category $\mathfrak{DT}_{fcq}$ having for objects the differential triads $\vec{\mathfrak{T}}_i$ and for arrows the finitary analogues of the triad-morphisms mentioned above is a subcategory of $\mathfrak{DT}$ called *the category of fcq-differential triads*. So, we let $\overset{\rightleftarrows}{\mathcal{T}} := \{\vec{\mathfrak{T}}_i\}$ be the *mixed projective-inductive* system of fcq-differential triads in $\mathfrak{DT}_{fcq}$.[123] By straightforwardly applying Papatriantafillou's results [81, 82] to the inverse-direct system $\overset{\rightleftarrows}{\mathcal{T}}$ we obtain a projective-inductive limit triad $\mathfrak{T}_\infty = (\mathbf{A} \equiv {}^{(\mathbb{K})}\mathcal{C}^\infty_X, \mathbf{\Omega}^p_\infty, d^p_\infty)$ (write: $\mathfrak{T}_\infty = \underset{\longleftarrow}{\overset{\longrightarrow}{\lim}} \overset{\rightleftarrows}{\mathcal{T}} \equiv \underset{\infty \leftarrow i}{\overset{i\rightarrow\infty}{\lim}} \{\vec{\mathfrak{T}}_i\}$), here called '$\mathcal{C}^\infty$-`smooth` differential triad', consisting of the structure

mon, fixed base topological space X (write $\mathfrak{T}_i(X)$), or a projective/inductive system thereof indexed by the same set of indices (write $\mathfrak{T}_i(X_i)$), possess projective/inductive limits. Below, we will see that our projective/inductive system $\overleftarrow{\mathcal{G}} = \{(\vec{\mathcal{P}}_i^{\uparrow}, \vec{\mathcal{D}}_i)\}$ of finitary posets (causets), (principal) finsheaves of incidence algebras (qausets) over them and reticular spin-Lorentzian connections on those finsheaves, are precisely of the second kind. The reader should also note here that in the mathematics literature, 'projective', 'inverse' and 'categorical' limits are synonymous terms; so are 'inductive' and 'direct' limits (also known as 'categorical colimits'). The result from [82] quoted above can be stated as follows: *the category* $\mathfrak{DT}$ *is complete and cocomplete*. This remark, that is to say, that $\mathfrak{DT}$ is (co)complete, will prove to be of great importance in current research [91] for showing that the category of finsheaves of qausets—which is a subcategory of $\mathfrak{DT}$—is, in fact, an example of a structure known as a *topos* [64]—a topos with a non-Boolean (intuitionistic) internal logic tailor-made to suit the finitary, causal and quantal vacuum Einstein-Lorentzian gravity developed in the present paper.

[121] From a categorical point of view, this fact alone suffices for regarding the abstract differential spaces (of structure sheaves of generalized differential algebras of functions and differential modules over them) that the ADG-theoretic differential triads represent as being more powerful and versatile differential geometric objects than $\mathcal{C}^\infty$-manifolds. As also mentioned in [82], it was precisely due to the aforesaid shortcomings of the category of smooth manifolds that led many authors in the past to generalize differential manifolds to *differential spaces* in which the manifold structure is effectively redundant [108, 109, 80, 51]. In fact, the first author's differential triads generalize both $\mathcal{C}^\infty$-manifolds and differential spaces, and, perhaps more importantly for the physical applications, they are general enough to include non-smooth ('singular') spaces with the most general, non-functional, structure sheaves [77, 78, 97]. On the other hand, a little bit later we will allude to and, based on ADG and its finitary application herein, comment on an example from [8] of an inverse system of differential manifolds that yields a differential manifold at the projective limit.

[122] With $(\mathcal{I}, \geq)$ a partially ordered, directed set (net) of indices 'i' labelling the elements of the inverse/direct system $(\mathcal{P}_i, \dot{\mathcal{D}}_i)$. The systems $(\mathcal{P}_i, \dot{\mathcal{D}}_i)$ are said to be (co)final with respect to the index net $(\mathcal{I}, \geq)$. We remind the reader that in our case 'i' is the finitarity or localization index (*ie*, locally finite open covers $\mathcal{U}_i$ of $X \subset M$ form a net [114, 87, 74, 75]).

[123] The term 'mixed projective-inductive' (or equivalently, '*mixed inverse-direct*') system pertains to the fact that the family $\overset{\rightleftarrows}{\mathcal{T}}$ (implicitly) contains both the projective system $\overleftarrow{\mathcal{P}} = \{\vec{P}_i\}$ of reticular base causets, and the inductive system $\overrightarrow{\mathfrak{R}}$ of qausets corresponding (by Gel'fand duality) to the aforesaid causets. (Note that we refrain from putting right-pointing causal arrows over $\overleftarrow{\mathcal{P}}$ and $\overrightarrow{\mathfrak{R}}$, in order to avoid notational confusion.)

sheaf $\mathcal{C}^{\infty}_X$ of generalized infinitely differentiable (*ie*, $\mathcal{C}^{\infty}$-`smooth`) functions on X, as well as of (sheaves $\mathbf{\Omega}^p_{\infty}$ over X of) ${}^{(\mathbb{K})}\mathcal{C}^{\infty}(X)$-bimodules Ω^p_{∞} of $\mathbb{K}$-valued differential forms related by exterior differentials ($\mathbf{K}$-linear sheaf morphisms) d^p_{∞}.

We can then localize or gauge the Cartan-Kähler differentials of the fcq-differential triads in $\mathfrak{D}\mathfrak{T}_{fcq}$ as worked out in [74], thus obtain the inverse system $\overleftarrow{\mathcal{G}} = \{(\vec{\mathcal{P}}^{\uparrow}_i, \vec{\mathcal{D}}_i)\}$ alluded to above.[124] As mentioned earlier, the limits of projective systems of principal sheaves equipped with Mallios $\mathbf{A}$-connections have been established in [124, 125, 126]. Hence, by straightforwardly carrying Vassiliou's results to the finitary case, and as it was anticipated in [74, 75], we get that $\overleftarrow{\mathcal{G}}$ yields at the projective limit a generalized classical principal $\mathcal{C}^{\infty}$-`smooth` (spin-Lorentzian) fiber bundle (whose associated bundle is the $\mathcal{C}^{\infty}$-`smooth` (co)tangent vector bundle of ${}^{(\mathbb{K})}\mathcal{C}^{\infty}(X)$-modules of $\mathbb{K}$-valued differential forms) endowed with a `smooth` $so(1,3)^{\uparrow}$-valued connection 1-form $\mathcal{A}$ over a (region X of) the $\mathcal{C}^{\infty}$-smooth spacetime manifold M[125] [74, 75]. All in all, we formally write

$$\begin{gathered}\mathfrak{T}_{\infty} = (\mathbf{A}_X \equiv {}^{\mathbb{K}}\mathcal{C}^{\infty}_X, \mathbf{\Omega}^p_{\infty}, d^p_{\infty}) = \underleftarrow{\overrightarrow{\lim}}\, \overleftrightarrows{\mathcal{T}} \equiv \lim_{\infty\leftarrow i}^{i\to\infty}\{\vec{\mathfrak{T}}_i\} \equiv \lim_{\infty\leftarrow i}^{i\to\infty}\{(\vec{\mathbf{A}}_i, \vec{\mathbf{D}}_i, \vec{d}^p_i)\} \\ ({}^{(\mathbb{K})}\vec{\mathcal{P}}_{\infty}, {}^{(\mathbb{K})}\mathcal{D}_{\infty}) = \underleftarrow{\lim}\, \overleftarrow{\mathcal{G}} \equiv \lim_{\infty\leftarrow i}\{(\vec{\mathcal{P}}^{\uparrow}_i, \vec{\mathcal{D}}_i)\}\end{gathered} \tag{115}$$

and diagrammatically one can depict these limiting procedures as follows

$$\begin{tikzcd}[column sep=huge]
\vec{\mathfrak{T}}_i \arrow[r, "\text{gauging}", "\vec{\partial}_i \longrightarrow \vec{\mathcal{D}}_i = \vec{\partial}_i + \vec{\mathcal{A}}_i"'] \arrow[d, "\text{injective tri}-"', "\text{ad morphism}"] & (\vec{\mathcal{P}}^{\uparrow}_i, \vec{\mathcal{D}}_i) \arrow[d, "\text{injective } \mathcal{G}_{\mathrm{i}}-\text{fin}-"', "\text{sheaf morphism}"] \\
\vec{\mathfrak{T}}_j \arrow[r, "\text{gauging}", "\vec{\partial}_j \longrightarrow \vec{\mathcal{D}}_j = \vec{\partial}_j + \vec{\mathcal{A}}_j"'] & (\vec{\mathcal{P}}^{\uparrow}_j, \vec{\mathcal{D}}_j) \\
\vdots \arrow[d, "\text{infinite}"', "\text{refinement}"] & \vdots \arrow[d, "\text{infinite}"', "\text{refinement}"] \\
\mathfrak{T}_{\infty} = \underleftarrow{\overrightarrow{\lim}}\, \overleftrightarrows{\mathcal{T}} \arrow[r, "\text{gauging}", "\partial_{\infty} \longrightarrow \mathcal{D}_{\infty} = \partial_{\infty} + \mathcal{A}_{\infty}"'] & ({}^{(\mathbb{K})}\mathcal{P}^{\uparrow}_{\infty}, {}^{(\mathbb{K})}\mathcal{D}_{\infty}) = \lim_{\infty\leftarrow i} \overleftarrow{\mathcal{G}}
\end{tikzcd} \tag{116}$$

4.2.1 A brief note on projective versus inductive limits

We mentioned earlier the categorical duality between the category $\mathfrak{P}$ of finitary substitutes P_i and poset morphisms between them, and the category $\mathfrak{Z}$ of the incidence algebras Ω_i associated with the P_is and injective algebra homomorphisms between them, which duality is ultimately rooted in

[124] We could have chosen to present the collection $\{(\vec{\mathcal{P}}^{\uparrow}_i, \vec{\mathcal{D}}_i)\}$ as an inductive family of principal finsheaves and their finitary connections, since the connections (of any order p) $\vec{\mathcal{D}}^p_i$ in each of its terms are effectively obtained by localizing or gauging the reticular differentials $\vec{d}^p_i$ in each term of $\overrightarrow{\mathfrak{R}}$. However, that we present $\overleftarrow{\mathcal{G}}$ dually, as an inverse system, is consistent with our previous work [74, 75] and, as we shall see shortly, it yields the same result at the continuum limit (*ie*, the $\mathcal{C}^{\infty}$-principal bundle).

[125] Write $({}^{(\mathbb{K})}P_{\infty}, {}^{(\mathbb{K})}\mathcal{D}_{\infty})$ for the $\mathcal{C}^{\infty}$-`smooth` principal bundle and its non-trivial spin-Lorentzian connection.

the general notion of Gel'fand duality.[126] In a topological context, the idea to substitute Sorkin's finitary topological posets by incidence Rota algebras was originally aimed at *'algebraizing space'* [129]—that is to say, at replacing 'space' (of which, anyway, we have no physical experience[127]) by suitable (algebraic) objects that may be perceived as living on that 'space' and, more importantly, from which objects this 'space' may be somehow derived by an appropriate procedure (Gel'fand spatialization). In fact, as briefly described before, again in a topological context and in the same spirit of Gel'fand duality, the second author substituted Sorkin's P_is by finsheaves S_i of (algebras of) continuous functions that, as we said, are (locally) topologically equivalent (*ie*, locally homeomorphic) spaces to the P_is [87]. Here too, the basic idea was, in an operational spirit, to replace 'space' by suitable algebraic objects that live on 'it', and it was observed that the maximum localization (finest resolution) of the point-events of the bounded region X of the $\mathcal{C}^0$-spacetime manifold M by coarse, open regions about them at the inverse limit of a projective system of P_is, corresponds (by Gel'fand duality) to defining the stalks of $\mathcal{C}^0_X$—the sheaf of (germs of) continuous functions on the topological manifold X—at the *direct limit* of (infinite localization of) *an inductive system of the* S_is.[128] At the end of [87] it was intuited that if the stalks of the S_is were assumed to be inhabited by incidence algebras which are discrete differential manifolds as explained above, at the inverse limit of infinite refinement or localization of the projective system $\overleftarrow{\mathcal{P}}$ of Sorkin's topological posets yielding the continuous base topological space X, the corresponding (by Gel'fand duality) inverse-direct system $\overset{\rightleftarrows}{\mathfrak{T}}$ of finitary differential triads should yield the classical structure sheaf $\mathbf{A}_X \equiv {}^{(\mathbb{C})}\mathcal{C}^\infty_X$ of germs of sections of (complex-valued)[129] smooth functions on X and the sheaf ${}^{\mathbb{C}}\mathbf{\Omega}_X$ of ${}^{(\mathbb{C})}\mathcal{C}^\infty(X)$-bimodules of (complex) differential forms, in accordance with Gel'fand duality.

There are two issues to be brought up here about this intuition at the end of [87]. First thing to mention is that, as alluded to earlier, it is more accurate to say that, since the incidence algebras are objects categorically or Gel'fand dual to Sorkin's topological posets, and since the latter form an inverse or projective system $\overleftarrow{\mathcal{P}}$, the former should be thought of as constituting a direct or inductive system $\overrightarrow{\mathfrak{R}}$ of algebras possessing ${}^{\mathbb{K}}\mathcal{C}^\infty(X)$ and ${}^{(\mathbb{K})}\Omega(X)$ over it as an *inductive limit.*[130] In fact, as mentioned in the previous paragraph, the stalks of ${}^{(\mathbb{K})}\mathbf{\Omega}_X$ (in fact, of any sheaf! [87]), which are inhabited by germs of sections of $\mathcal{C}^\infty$-smooth ($\mathbb{K}=\mathbb{R},\mathbb{C}$-valued) differential forms, are obtained precisely at that inductive limit. We may distill all this to the following physical statement which foreshadows our remarks on Gel'fand duality to be presented in the next section:

> *While 'space(time)' is maximally (infinitely) localized (to its points) by an inverse limit of a projective system of Sorkin's finitary posets, the (algebraic) objects that live*

[126] See our more analytical comments on Gel'fand duality in the next section.

[127] Again, see more analytical comments on the 'unphysicality' of space(time) in the next section.

[128] And it should be emphasized that the stalks of a sheaf are the 'ultra-local' (*ie*, maximally localized) point-like elements of the sheaf space [67, 87].

[129] In [92, 86, 87, 93, 74] it was tacitly assumed that we were considering incidence algebras over the field $\mathbb{C}$ of complex numbers.

[130] Hence, precisely speaking, the aforesaid fcq-differential triads constitute a mixed inverse-direct system $\overset{\rightleftarrows}{\mathfrak{T}}$ having the $\mathcal{C}^\infty$-`smooth` differential triad $\mathfrak{T}_\infty$ as an inductive limit [81, 82].

on space(time) (ie, the various physical fields) are maximally (infinitely) localized in the stalks of the finsheaves that they constitute by a direct limit of an inductive system of those finsheaves. Equivalently stated, *'space(time)' is categorically or Gel'fand dual to the physical fields that are defined on 'it'.*

The second thing that should be stressed here, and in connection with footnote 75, is that we do not actually get the classical differential geometric structure sheaf ${}^{\mathbb{K}}\mathcal{C}^{\infty}_X$ and the corresponding sheaf ${}^{\mathbb{K}}\mathbf{\Omega}_X$ of ${}^{\mathbb{K}}\mathcal{C}^{\infty}(X)$-modules of differential forms. *In toto*, we do not actually recover the classical $\mathcal{C}^{\infty}$-smooth differential triad $\mathtt{T}_{\infty} := (\mathbf{A}_X \equiv {}^{\mathbb{K}}\mathcal{C}^{\infty}_X, \partial, \mathbf{\Omega}^1_X)$ at the limit of infinite localization of the system $\overset{\rightleftarrows}{\mathcal{T}}$, but rather we get the *generalized smooth* (*ie*, what we call here $\mathcal{C}^{\infty}$-$\mathtt{smooth}$) triad $\mathfrak{T}_{\infty} = (\mathbf{A}_X \equiv {}^{\mathbb{K}}\mathcal{C}^{\infty}_X, \mathbf{\Omega}^p_{\infty}, d^p_{\infty})$. Of course, by the general theory (*ie*, ADG), we are guaranteed that the direct, cofinal system $\overset{\rightleftarrows}{\mathcal{T}}$ of 'generalized discrete differential spaces'—that is, the fcq-triads $\vec{\mathfrak{T}}_i = (\vec{\mathbf{A}}_i, \vec{\mathbf{D}}_i, \vec{d}^p_i)$—yields a well defined differential structure at the categorical colimit within $\mathfrak{DT}$; moreover, according to ADG, it is quite irrelevant whether the differential triad at the limit is the classical smooth $\mathtt{T}_{\infty}$ of the featureless $\mathcal{C}^{\infty}$-manifold proper or one for example that is infested by singularities thus most pathological and unmanageable when viewed from the classical $\mathcal{C}^{\infty}$-manifold perspective [77, 78, 97].[131] The point we wish to make here is simply that at the continuum limit we get *a*, not *the* familiar $\mathcal{C}^{\infty}$-smooth, differential structure on the continuous topological ($\mathcal{C}^0$) spacetime manifold X. This differential structure 'for all practical purposes' represents for us the classical, albeit 'generalized', differential manifold, and the direct limiting procedure that recovers it a generalized version of Bohr's correspondence principle advocated in [92]. That this differential structure obtained at the 'classical limit' is indeed adequate for accommodating the classical theory will become transparent in the next section where we will see that based on $\mathfrak{T}_{\infty}$ we can actually write the classical vacuum Einstein equations of general relativity; albeit, in a generalized, ADG-theoretic way *independently of the usual $\mathcal{C}^{\infty}$-manifold.* In fact, we will see that these equations are obtained at the inverse limit of a projective system $\overleftarrow{\mathcal{E}}$ of vacuum Einstein equations—one for each member of $\overset{\rightleftarrows}{\mathcal{T}}$.

4.2.2 Some comments on real versus complex spacetime and the general use of the number fields $\mathbb{R}$ and $\mathbb{C}$

As it has been already anticipated in [74, 90], starting from principal finsheaves of *complex* ($\mathbb{K} = \mathbb{C}$) incidence algebras carrying non-flat reticular spin-Lorentzian $\vec{\mathbf{A}}_i$-connections $\vec{\mathcal{D}}_i$ as $\mathbf{C}$-linear finsheaf morphisms between the 'discrete' differential $\vec{\mathbf{A}}_i$-bimodules $\vec{\mathbf{\Omega}}^p_i$ ($p \geq 1$) in $\vec{\mathbf{D}}_i$, *complex* (bundles of) $\mathtt{smooth}$ coordinate algebras, modules of differential forms over them[132] and $\mathtt{smooth}$ $so(1,3)^{\uparrow}_{\mathbb{C}}$-valued connection 1-forms $\mathcal{A}$ (over a $\mathtt{smooth}$ complex manifold) are expected to emerge at the inductive/projective limit of infinite refinement and localization of the qausets and the principal finsheaves thereof.[133] Thus it may be inferred that in order to recover the real

[131] See footnote 121.

[132] That is to say, the generalized 'classical', $\mathcal{C}^{\infty}$-$\mathtt{smooth}$ differential triad $\mathfrak{T}_{\infty}$ mentioned above.

[133] Indeed, in the context of non-perturbative (canonical) quantum gravity using Ashtekar's new gravitational connection variables, we will see in the next section how a holomorphic Lorentzian spacetime manifold and smooth,

spacetime continuum of macroscopic relativistic gravity (general relativity), some sort of *reality conditions* must be imposed after the projective limit, the technical details of which have not been fully investigated yet [130, 131]. The nature of these conditions is a highly non-trivial and subtle issue in current quantum gravity research [14].

On the other hand, starting from incidence algebras over $\mathbb{R}$ ($\mathbb{K} = \mathbb{R}$), one should be able recover a *real* $\mathcal{C}^{\infty}$-`smooth` manifold instead of a complex one at the projective/inductive classical limit', but then one would not be faithful to the conventional quantum theory with its continuous coherent superpositions over $\mathbb{C}$.[134] On the other hand, *prima facie* it appears to be begging the question to maintain that we have an 'innately' or 'intrinsically finitistic' model for the kinematical structure of Lorentzian quantum spacetime and gravity (and, as we shall contend in the following section, also for the dynamics) when its (noncommutative) algebraic representation employs *ab initio* the continuum of complex numbers as the field of (probability) amplitudes.

For example, in the light of application of ideas from presheaves and topos theory to quantum gravity, Butterfield and Isham [21], and more recently Isham [55], have also explicitly doubted and criticized the *a priori* assumption and use of the continuum of either the reals or, *a fortiori*, of the complexes in quantum theory *vis-à-vis* the quest for a genuinely quantum theoresis of spacetime structure and gravity. In [55] in particular, Isham maintains that the use of the arithmetic continua of $\mathbb{R}$ (modelling probabilities and the values of physical quantities) and $\mathbb{C}$ (probability amplitudes) in standard quantum mechanics is intimately related (in fact, ultimately due) to the *a priori* assumption of a classical stance against the 'nature' of space and time—*ie*, the assumption of the classical spacetime continuum. In the sequel, in order to make clear-cut remarks on this in relation to ADG, as well as to avoid as much as we can 'vague dark apostrophes', by 'spacetime continuum' we understand the locally Euclidean arena (*ie*, the manifold) that (macroscopic) physics uses up-front to model spacetime. Our contention then is that *Isham questions the use of $\mathbb{R}$ and $\mathbb{C}$ in quantum theory precisely because he is motivated by the quest for a genuinely quantum theoresis of spacetime and gravity*, for in quantum gravity research it has long been maintained that the classical spacetime continuum (*ie*, the manifold) must be abandoned in the sub-Planckian regime where quantum gravitational effects are expected to be significant.[135] Thus, his basic feeling is that the conventional quantum theory, with its continuous superpositions over $\mathbb{C}$ and probabilities in $\mathbb{R}$, which it basically inherits from the classical spacetime manifold, must be modified *vis-à-vis* quantum gravity. *In toto*, *if the manifold has to go in the quantum deep, so must the number fields $\mathbb{R}$ and $\mathbb{C}$ of the usual quantum mechanics, with a concomitant relatively drastic modification of the usual quantum formalism to suit the non-continuum base*

complex (self-dual) connections on it are the basic dynamical elements of the theory.

[134] And indeed, in [86, 93, 75] the $\mathbb{C}$-linear combinations of elements of the incidence algebras where physically interpreted as *coherent quantum superpositions* of the causal-topological arrow connections between the event-vertices in the corresponding causets. In fact, it is precisely this $\mathbb{C}$-linear structure of the qausets that qualifies them as sound *quantum* algebraic analogues of causets, which are just associative multiplication structures (arrow semigroups or monoids or even poset categories). Also, in connection with footnote 87, we emphasize that it is the linear structure of qausets (prominently absent from causets) that gives them both their differential (geometric) and their quantum character.

[135] For instance, see the two opening quotations.

space(time).[136] Perhaps the use from the beginning of one of the finite number fields $\mathbb{Z}_p$[137] for c-numbers would be a more suitable choice for our reticular models, but then again, what kind of quantum theory can one make out of them?[138] The contents of this paragraph are captured nicely by the following excerpt from [55]:

> "*...These number systems* [*ie*, $\mathbb{R}$ and $\mathbb{C}$] *have a variety of relevant mathematical properties, but the one of particular interest here is that they are continua, by which—in the present context—is meant not only that $\mathbb{R}$ and $\mathbb{C}$ have the appropriate cardinality, but also that they come equipped with the familiar topology and differential structure that makes them manifolds of real dimension one and two respectively. My concern is that the use of these numbers may be problematic in the context of a quantum gravity theory whose underlying notion of space and time is different from that of a smooth manifold. The danger is that by imposing a continuum structure in the quantum theory a priori, one may be creating a theoretical system that is fundamentally unsuitable for the incorporation of spatio-temporal concepts of a non-continuum nature: this would be the theoretical-physics analogue of what a philosopher might call a 'category error'...*"

while, two years earlier [21], Butterfield and Isham made even more sweeping remarks about the use of smooth manifolds in physics in general, and their inappropriateness *vis-à-vis* quantum gravity:

> "*...the first point to recognise is of course that the whole edifice of physics, both classical and quantum, depends upon applying calculus and its higher developments (for example, functional analysis and differential geometry) to the values of physical quantities...why should space be modelled using $\mathbb{R}$? More specifically, we ask, in the light of* [our remarks above about the use of the continuum of the real numbers as the values of physical quantities]: *Can any reason be given apart from the (admittedly, immense) 'instrumental utility' of doing so, in the physical theories we have so far developed? In short, our answer is No. In particular, we believe there is no good a priori reason why space should be a continuum; similarly, mutatis mutandis for time. But then the crucial question arises of how this possibility of a non-continuum space should be reflected in our basic theories, in particular in quantum theory itself, which is one of the central ingredients of quantum gravity...*"[139]

At this point it must be emphasized that in ADG, $\mathbb{R}$ and $\mathbb{C}$ enter the theory through the generalized arithmetics—the structure sheaf $\mathbf{A}_X$, which, as noted earlier, is supposed to be a sheaf of commutative $\mathbb{K} = \mathbb{R}, \mathbb{C}$-algebras (*ie*, $\mathbf{K} = \mathbf{R}, \mathbf{C} \hookrightarrow \mathbf{A}$). In turn, these arithmetics are invoked only when one wishes to represent local measurements and do with them general

[136] See below.
[137] With 'p' a prime integer.
[138] Chris Isham in private communication.
[139] Excerpt from "*Whence the Continuum?*" in [21]. These remarks clearly pronounce our application here of ADG, which totally evades the usual $\mathcal{C}^\infty$-calculus, to finitary Lorentzian quantum gravity (see also remarks below).

calculations with the vector sheaves $\mathcal{E}$ employed by ADG.[140] It is at this point that the basic assumption of ADG that the $\mathcal{E}$s involved are locally free $\mathbf{A}$-modules of finite rank n–that is to say, locally isomorphic to $\mathbf{A}^n$—comes in handy, for all our local measurements and calculations involve $\mathbf{A}$, $\mathbf{A}^n$ and, *in extenso*, the latter's natural local transformation matrix group $\mathcal{A}ut\mathcal{E}(U) = \mathcal{E}nd\mathcal{E}(U)^\bullet \equiv M_n(\mathbf{A}(U))^\bullet$. Thus, *real and complex numbers enter our theory through 'the back-door of measurement and calculation'*, *in toto*, through 'geometry' as understood by ADG.[141]

On the other hand, and in connection with the last footnote, since the constructions of ADG are genuinely independent of (the usual calculus on) $\mathcal{C}^\infty$-manifolds,[142], whether real (analytic) or complex (holomorphic), [67, 68, 77, 78, 74, 70, 75, 73], Isham's remarks that the appearance of the arithmentic continua in quantum theory are due to the *a priori* assumption of a classical spacetime continuum—a locally Euclidean manifold—do not affect ADG. Of course, we would actually like to have at our disposal the usual number fields in order to be able to carry out numerical calculations (and arithmetize our abstract algebraic sheaf theory) especially in the (quantum) *physical* applications of ADG that we have in mind.[143] We may distill all this to the following:

> In the general ADG theory, and in its particular finitary application to quantum gravity here, the commutative number fields, which happen to be locally Euclidean continua (*ie*, the manifolds $\mathbb{R} \simeq \mathbb{R}^1$ and $\mathbb{C} \simeq \mathbb{R}^2$ being equipped with the usual differential geometric—*ie*, $\mathcal{C}^\infty$-smooth—structure), do not appear in the theory from assuming up-front a background spacetime manifold.[144] Rather, they are only built

[140] See sections 2 and 3, and in particular the discussion in 4.3 next.

[141] This is in accord with our view of $\mathbf{A}$ mentioned earlier as the structure carrying information about the 'geometry', about our own measurements of 'it all' (see footnotes 17, 41, the end of 2.3 and subsection 4.3 next). In agreement with Isham's remarks in [55] briefly mentioned above, *it is we, with our classical manifold conception of space and time, who bring $\mathbb{R}$ and $\mathbb{C}$ into our models of the quantum realm.* The quantum deep itself has no 'numbers' as such, and it is only our observations, measurements—in effect, 'geometrizations—of 'it all' that employs such c-numbers (Bohr's correspondence principle). *Nature has no number or metric; we dress Her in such, admittedly ingenious, artifacts* (see footnote 17 and also the following one). Based on ADG and its finitary application to Lorentzian quantum gravity here, shortly we will go a step further than Isham and altogether question the very notion of 'spacetime' in the quantum realm. Thus, under the prism of ADG, the question whether spacetime is 'classical' or 'quantum' should be put aside and the doubts of using $\mathbb{R}$ and $\mathbb{C}$ in quantum theory should not be dependent in any way on the answer to that question.

[142] In fact, of any 'background spacetime structure', whether 'continuous' or 'discrete'.

[143] For recall Feynman: "*The whole purpose of physics is to find a number, with decimal points etc. Otherwise, you haven't done anything.*" [39]—and arguably, numbers are obtained by measurements, observations and the general 'instrumental/operational-geometrical activity' that physicists exercise (in their local laboratories, 'with clocks and rulers' so to speak) on Nature. *Numbers are not Nature's own.* Thus, both the arithmetics, as encoded in the abelian algebra structure sheaf $\mathbf{A}$, and the $\mathbf{A}$-metric ρ relative to it, lie on the observer's (*ie*, the classical) side of the quantum divide and are not 'properties' of quantum systems—they are our own 'devices' (see footnote 17). This brings to mind Aeschylus' remark in 'Prometheus Bound': "*Number, the most ingenious of human inventions*" [1] (notwithstanding of course the innumerable modern debates among the philosophers of mathematics whether 'number' is a creation of the human mind or whether it exists, in a non-physical Platonic world of Ideas, 'out there').

[144] For, as we have time and again emphasized in this paper, ADG evades precisely this: doing the usual differential geometry (calculus) on a classical $\mathcal{C}^\infty$-smooth background manifold [67, 68, 74, 75].

into our generalized arithmetic algebra sheaf $\mathbf{A}_X$, thus they are of sole use in our local calculations and 'physical geometrization' (*ie*, 'analysis of measurement operations') of the abstract algebraic theory. As such, they are not actually liable to Isham's criticism and doubts,[145] for ADG totally evades the base geometric spacetime manifold.

For instance, from our ADG-theoretic perspective, this independence of measurement from an 'ambient' spacetime continuum and its focus solely on the (physical) objects (fields) *per se* that live on that background 'space(time)'—and perhaps more importantly, regardless of whether the latter is a 'discrete' or a continuous manifold base arena—may be seen as a 'post-anticipation' of Riemann's words in [96] which we quote *verbatim* from [73]:

> "*Maß bestimungen erfordern eine Unabhängichkeit der Größen vom Ort, die in mehr als einer Weise stattfinden kann.*" : "*Specifications* [: measurements] *of mass require an independence of quantity from position, which can happen in more than one way.*"

Moreover, as we shall see subsequently, and in contrast to Isham, we do not aim for a non-continuum theoresis of spacetime (and gravity) in order to abolish the *a priori* use of $\mathbb{R}$ and $\mathbb{C}$ in the usual quantum theory[146] with a concomitant modification of the latter to suit the non-continuum base spacetime, for there is no (background) 'spacetime' (whether 'discrete' or 'continuous') as such in the quantum deep and in ADG the (structural) role played the base (topological) space is a (physically) atrophic, inactive, dynamically non-participatory one.

The last remark also prompts us to highlight from [55] another remark of Isham that is quite relevant to our present work:[147]

> "*The main conclusion I wish to draw from the discussion above is that a number of a priori assumptions about the nature of space and time are present in the mathematical formalism of standard quantum theory, and it may therefore be necessary to seek a major restructuring of this formalism in situations* [like for example those motivated by quantum gravity ideas[148]] *where the underlying spatio-temporal concepts (if there are any at all) are different from the standard ones which are represented mathematically with the aid of differential geometry.*[149]
>
> *A good example would be to consider from scratch how to construct a quantum theory when space-time is a finite causal set: either a single such—which then forms a fixed, but non-standard, spatio-temporal background—or else a collection of such sets in the context*

[145] That is, again, that the use of the fields of real (probabilities) and complex (probability amplitudes) numbers in quantum theory is basically due to the *a priori* assumption of a classical spacetime manifold.

[146] At least, as long as we abide to the operational idea that our quantal operations, which classically involve (ideal) clocks and measuring rods [38, 112, 48] which, in turn, are admittedly modelled after $\mathbb{R}$ [55], are organized into (noncommutative) algebras (*ie*, in line with Heisenberg's conception of an algebraically implemented 'quantum operationality' [74]) as well as that upon measurement they yield commutative numbers in the base field (Bohr).

[147] The excerpt below is taken from section 2.2 in [55] titled '*Space-time dependent quantum theory*'.

[148] Our addition to tie the text with what Isham was discussing prior to it.

[149] And, of course, Isham refers to the usual differential geometry of $\mathcal{C}^\infty$-manifolds.

of a type of quantum gravity theory. In the case of a fixed background, this new quantum formalism should be adapted to the precise structure of the background, and can be expected to involve a substantial departure from the standard formalism: in particular to the use of real numbers as the values of physical quantities and probabilities."

In the next section we will see exactly how, with the help of ADG, we can write the vacuum Einstein equations for Lorentzian gravity over a causet and, in contradistinction to Isham's remarks above, without having to radically modify quantum theory—in particular, in its use of $\mathbb{R}$ and $\mathbb{C}$—in order to suit that discrete, non-continuum background spacetime. As a matter of fact, we will see that this base causet plays no physically significant role apart from serving as a (fin)sheaf-theoretic localization scaffolding in our theory; moreover, no quantum theory proper (either the standard one, or a modified one intuited by Isham above) will be employed to quantize the classical theory (*ie*, Einstein's equations on the smooth manifold). All in all, as we will witness in the sequel, in a strong sense our ADG-based finitary vacuum Einstein gravity may be perceived as being 'inherently' or 'already quantum', 'fully covariant'—*ie*, as involving only the dynamical fields and not being dependent in any way on an external, base spacetime, be it granular or a smooth continuum, and certainly as not being the outcome of applying quantum theory (*ie*, 'formally quantizing') the classical theory of gravity on a spacetime manifold (*ie*, general relativity).

4.2.3 A brief 'critique' of the Ashtekar-Lewandowski projective limit scheme

In [8], a projective system $\overleftarrow{\mathcal{M}}$ of compact Hausdorff manifolds labelled by graphs—which can be physically interpreted as 'floating lattices'—was employed in order to endow, at the projective limit of that family of manifolds, the moduli space $\mathcal{A}_\infty/\mathcal{G}$ of $\mathcal{C}^\infty$-smooth gauge ($\mathcal{G}$)-equivalent Y-M or (self-dual) gravitational connections with a differential geometric structure including vector fields, differential forms, exterior derivatives, metric volume forms, Laplace operators and their measures, as well as the rest of the familiar $\mathcal{C}^\infty$-smooth differential geometric entities. As we shall see in the next section, there has been an ever growing need in current approaches to non-perturbative canonical (Hamiltonian, loop variables-based) or covariant (Lagrangian, action-based) quantum gravity, to acquire a firm tangent bundle perspective on $\mathcal{A}_\infty/\mathcal{G}$ (*ie*, have a mathematically well defined $T(\mathcal{A}_\infty/\mathcal{G})$ object), since $T(\mathcal{A}_\infty/\mathcal{G})$ can serve as the physical phase space of quantum Y-M theories and gravity in its gauge-theoretic form in terms of Ashtekar's (self-dual) connection variables [3] and one would like to do differential geometry on that space. Thus, the basic idea is that if such a mathematically rigorous differential geometric status is first established on the moduli space, one could then hope to tackle deep quantum gravity problems such as the Hilbert space inner product (and measure) problem, the problem of time, the non-trivial character of $\mathcal{A}_\infty/\mathcal{G}$ when regarded as a $\mathcal{G}$-bundle, the problem of physical Wilson loop observables *etc*[150] by the conventional calculus-based (*ie*, the usual $\mathcal{C}^\infty$-differential geometric) methods of the canonical or the covariant approaches to quantum field theory.

Although, admittedly, algebraic methods were used in [8] towards endowing the moduli space of connections with the conventional differential geometric apparatus, the very nature

[150]See 5.3 for a more analytical exposition and discussion of some of these problems.

(*ie*, the $\mathcal{C}^\infty$-smooth character) of each member of $\overleftarrow{\mathcal{M}}$ shows the original intention of Ashtekar and Lewandowski: in order to induce the usual $\mathcal{C}^\infty$-differential geometric structure on $\mathcal{A}_\infty/\mathcal{G}$ at the projective limit, one must secure that each member of the inverse system $\overleftarrow{\mathcal{M}}$ comes equipped with such a structure—that is to say, it is a differential manifold itself. In other words, as it was already mentioned in the beginning of this section, the essence of [8] is that *like differential structure yields* (*ie*, induces at the inverse limit) *like differential structure*. Now, in view of the fact that some (if not all!) of the aforementioned problems of $T(\mathcal{A}_\infty/\mathcal{G})$ come precisely from the $\mathcal{C}^\infty$-smoothness of the spacetime manifold and, concomitantly, from the group $\mathrm{Diff}(M)$ of its 'structure symmetries'[151], it appears to us that this endeavor is to some extent 'begging the question'.[152] Of course, it is quite understandable with 'general relativity or $\mathcal{C}^\infty$-smooth spacetime manifold-conservative' approaches to quantum gravity, such as the canonical or the covariant (path-integral) ones,[153] to maintain that the differential geometric mechanism is intimately tied to (or comes from) the differential manifold, for, after all, *manifolds were created for the tangent bundle.*[154]

However, this is precisely the point of ADG: the intrinsic, 'inherent' mechanism of differential geometry has nothing to do with $\mathcal{C}^\infty$-smoothness, nothing to do with $\mathcal{C}^\infty$-smooth manifolds, and the latter (in fact, its structure sheaf $\mathcal{C}^\infty_M$) provide us with just a (the classical, and by no means *the* 'preferred', one) 'mechanism for differentiating'.[155] For instance, as we saw in sections 2 and 3, one can develop a full-fledged differential geometry, entirely by algebraic (*ie*, sheaf-theoretic) means and completely independently of $\mathcal{C}^\infty$-smoothness, on the affine space of connections as well as on the moduli space of gauge-equivalent connections.[156] In the finitary case of interest here, and in striking contradistinction to [8], we have seen above (and in the past [74, 75]) how each principal finsheaf $\vec{\mathcal{P}}_i^\uparrow$ of qausets in the projective system $\overleftarrow{\mathsf{G}}$ carries virtually all the differential geometric panoply without being dependent at all on the classical $\mathcal{C}^\infty$-manifold. In fact, in the next section we will see how such a $\mathcal{C}^\infty$-smooth spacetime manifold-free scenario will not prevent us at all from writing a locally finite version of the usual Einstein equations for vacuum Lorentzian gravity. Quite on the contrary, it will enable us to evade altogether $\mathrm{Diff}(M)$ as well as some of the aforesaid problems that the latter group creates in our search for a cogent non-perturbative quantum gravity, whether canonical or covariant, on the moduli space of gravitational connections. Moreover, we will see how we can recover the $\mathtt{C}^\infty$-$\mathtt{smooth}$ vacuum Einstein equations at the projective limit of an inverse system $\overleftarrow{\mathsf{E}}$ of fcqv-ones. Already at a kinematical level, at the end of the next subsection we will argue ADG-theoretically how the 'generalized classical' $\mathtt{C}^\infty$-$\mathtt{smooth}$ moduli space of gauge-equivalent (self-dual) spin-Lorentzian connections can be obtained at the inverse limit of an inverse system $\overleftarrow{\mathsf{M}}$ of fcqv-moduli spaces.

[151] See next section.

[152] The quest(ion) being for (about) a quantum gravitational scheme that is finitistic, but more importantly, *genuinely background $\mathcal{C}^\infty$-smooth spacetime manifold-free* (see following section).

[153] See category 1 in the prologue to this paper.

[154] In the next section we will return to comment further on this in connection with (140).

[155] See the concluding section about 'the relativity of differentiability'.

[156] For the full development of differential geometry *à la* ADG on gauge-theoretic moduli spaces, the reader is referred to [71].

But before we do this, let us recapitulate and dwell a bit longer on some central kinematical ideas that were mentioned *en passant* above.

4.3 Remarks on the 'Operational' Conception of Finitary Quantum Causality: a Summary of Key Kinematical Notions for Finitary, Causal and Quantal Vacuum Einstein-Lorentzian Gravity

Our main aim in this subsection is to highlight the principal new kinematical notions, of a strong operational-algebraic flavour, about 'finitary causality' originally introduced in [74]. In this way, we are going to emphasize even more the characteristic contrast between our '*operational and quantal*'—in fact, '*observer-dependent*'—conception of locally finite causality via qausets, and Sorkin *et al.*'s more '*realistic*' causet theory proper. As a main source for drawing this comparison of our approach against causet theory we are going to use [115]. Also, by this review we hope to make clearer to the reader the intimate connection between central ADG-theoretic notions such as 'open gauge', 'structure sheaf of generalized arithmetics/coordinates or measurements' *etc*, and some primitive notions of the finitary approach to spacetime (topology) as initially presented by Sorkin in [114].

With [74] as our main reference and compass to orientate us in this short review, we provide below a list of primitive assumptions, already explicitly or implicitly made in [114], that figure prominently in all our ADG-based trilogy (*ie*, in [74, 75] and here) on finitary spacetime and Lorentzian quantum gravity:

1. The basic intuitive and heuristic assumption is the following identification we made in [74]:

$$\text{“(coarse) localization”} \equiv \text{“(coarse) measurement/observation”} \tag{117}$$

   For the moment, assuming with Sorkin that topology is a 'predicate' or property of the (quantum) physical system 'spacetime', in the sense that "*the points of the manifold are the carriers of its topology*" [114], *we model our coarse measurements of (the topological relations between) spacetime point-events by 'regions' or 'open sets' about them.* Conversely, the open sets of a covering separate or distinguish the points of X. We thus have, for a bounded region X of a classical $\mathcal{C}^0$-spacetime manifold M,[157] and a locally finite open cover $\mathcal{U}_i$ of it,[158]

$$\text{“(coarse) determination of } x \in X\text{”} \equiv \text{“open set } U \in \mathcal{U}_i \text{ about } x\text{”} \tag{118}$$

[157] As explained in [92], the assumption of a bounded spacetime region X rests on the fact that actual or 'realistic' experiments are carried out in laboratories of finite size and are of finite duration.

[158] Again, as explained in [92], the assumption of a locally finite open covering $\mathcal{U}_i$ rests on the experimental fact that we always record, coarsely, a finite number of events.

2. Operationally speaking, it is widely recognized that *localization involves 'microscopic energy', and measurement a gauge.* We thus identify again (nomenclature-wise)

$$\text{“open set } U \in \mathcal{U}_i \text{ about } x\text{”} \equiv \text{“open gauge } U \text{ of } x\text{”} \tag{119}$$

   and note that this—*ie*, '*open gauge*'— is precisely the name ADG gives to the sets of the open coverings of the base topological or localization space X involved in a differential triad [67, 68].

3. Of course, the better (*ie*, more accurate or sharp) the localization, the higher the microscopic energy of resolution (of X into its point-events). Thus, we suppose that the locally finite open coverings of X form an inverse system or net (*ie*, a partially ordered set itself) with respect to the relation '$\succeq$' of *fine graining.* Roughly, better (more accurate or sharper) localization of x involves smaller and more numerous open sets about it, thus higher microscopic energy of resolution.

4. With these operational assumptions, Sorkin's two main results in [114] can be interpreted then as follows:

   — i) Sorkin's 'algorithm'—*ie*, the extraction of a T_0-topological poset P_i from X relative to a locally finite open cover $\mathcal{U}_i$—involves separating and grouping together into equivalence classes (of 'observational indistinguishability') the point-events of X relative to the open gauges U in $\mathcal{U}_i$.[159] *Point-events in the same equivalence class (which is a vertex) in P_i are interpreted as being indistinguishable relative to our coarse measurements or 'observations' in $\mathcal{U}_i$*, and

   — ii) Sorkin's inverse limit of the projective system of topological posets $\overleftarrow{\mathcal{P}}$ can now be interpreted as the recovering of the locally Euclidean $\mathcal{C}^0$-topology of X at the finest resolution or 'ultra localization' of X into its point-events. *In this sense, the continuous manifold topology is, operationally speaking, an ideal or 'non-pragmatic'* [92] *situation involving infinite (microscopic) energy of localization or measurement.*

5. Then came Sorkin's radical re-interpretation of the locally finite partial orders involved from topological to causal [115], which essentially planted the seed for causet theory. We recall from [115] a telling account of this conceptual sea-change:

   > "*...Still, the order inhering in the finite topological space seemed to be very different from the so-called causal order defining past and future. It had only a topological meaning but not (directly anyway) a causal one. In fact the big problem with the finite topological space was that it seemed to lack the information which would allow it to give rise to the continuum in all its aspects, not just in the topological aspect, but with its metrical (and therefore its causal) properties as well...The way out of the impasse involved a conceptual jump in which the formal mathematical structure remained constant, but*

[159]See [114, 86, 87, 74, 75] for more details about Sorkin's algorithm.

its physical interpretation changed from a topological to a causal one...The essential realization then was that, although order interpreted as topology seemed to lack the metric information needed to describe gravity, the very same order reinterpreted as a causal relationship, did possess information in a quite straightforward sense...In fact it took me several years to give up the idea of order-as-topology and adopt the causal set alternative as the one I had been searching for...

6. Now, the basic idea in [86], but most explicitly in [74] under the light of ADG, is that, in spite of Sorkin's semantic switch above, and in order to retain our picture of finitary posets as graded discrete differential manifolds (or homological objects/simplicial complexes),[160] we felt we had to give a more operational-algebraic (thus more easily interpretable quantum mechanically [86]) definition of finitary causality than causets. We read from [74] what this operational, observation $\mathcal{U}_i$-dependent conception of (quantum) causality involved:

> "...All in all, (quantum) causality is operationally defined and interpreted as a '*power relationship*' between spacetime events relative to our coarse observations (or approximate operations of local determination or 'measurement') of them, namely, if events x and y are coarsely determined by $\mathcal{N}(x)$ and $\mathcal{N}(y)$ with respect to $\mathcal{U}_i$, and $\mathcal{N}(y)\subset\mathcal{N}(x)$,[161] then we say that '*x causes y*'. The attractive feature of such a definition and interpretation of causality is that, by making it relative to $\mathcal{U}_i$, we render it '*frame-*' or '*gauge-*' or even '*observation-dependent*'..."[162]

Of course, the open sets in $\mathcal{U}_i$ now stand for '*coarse causal regions*' or rough operations of 'observation' or ' measurement' of the causal relations between events in the curved space*time* manifold [74], not just coarse approximations of the topological relations proper between events. Thus, in view of Sorkin's semantic switch quoted above from [115], as well as his assumption in [114] that the points of X are the carriers of its topology, we

[160] So that we could apply the differential geometric ideas of ADG, in a (fin)sheaf-theoretic context [87], at the reticular level of causets [75]. Indeed, the fundamental reason that we insist that the locally finite posets we are using are *simplicial complexes* is that the construction of the incidence algebras from such posets is manifestly *functorial* [92, 93, 130], which in turn secures that the (fin)sheaves over them *exist*. Had we, like Sorkin *et al.* insisted on *arbitrary* (locally finite) posets (see below), the correspondence 'finitary posets'⟶'incidence algebras' would not be functorial, and the (fin)sheaves that we would be talking about would not actually exist. Furthermore, the bonus from working with (locally finite) posets that are *a fortiori* simplicial complexes is that the (incidence algebras of the) latter, again as shown in [92, 93, 130], have a rich (discrete) graded differential structure, which has opened the possibility of applying ADG-theoretic ideas to the (fin)sheaves thereof.

[161] Where $\mathcal{N}(x)$ is effectively the Čech-Alexandrov 'nerve-cell' [22, 2] of x relative to $\mathcal{U}_i$, namely, the smallest open set $\bigcap\{U\in\mathcal{U}_i:\ x\in U\}$ in the subtopology of X (generated by countable unions of finite intersections of the open gauges U in $\mathcal{U}_i$) which includes x (see also [75]). By such cells one builds up (abstract) simplicial complexes (nerves) which, as noted before, are isomorphic to Sorkin's finitary T_0-topological posets in [114] essentially under two additional conditions on $\mathcal{U}_i$: that it is *generic* (*ie*, all non-trivial intersections of its open sets are different) and *minimal* (*ie*, if any of its open sets is omitted, it ceases being a covering of X) [93, 85]. (This footnote is not included in [74]).

[162] Such a cellular (simplicial), but more importantly to our physical interpretation here, 'coarse observation-dependent' ('perturbing operations-sensitive'), decomposition of spacetime, apart from Regge's celebrated paper [94], has been worked out by Cole [23] and very recently by [85]. (This footnote is also not included in [74]).

assume a more operational and at the same time less 'realistic' stance than Sorkin [115] by maintaining that *the point-events of X are the carriers of causality in relation to our coarse and perturbing observations (open gauges) U in $\mathcal{U}_i$* [74].

7. Having secured that our structures now enjoy both a causal and an operational interpretation, it became evident to us that our scheme differs fundamentally from Sorkin *et al.*'s causet scenario at least in the following two ways:

   — i) unlike the case in causet theory, which posits up-front a 'locally finite poset democracy', in our theoretical scheme not all locally finite posets and their incidence Rota algebras may qualify as being 'operationally sound qausets'. Only posets coming from coarse causal gauges $\vec{\mathcal{U}}_i$[163] and their incidence algebras are admissible as qausets proper. As mentioned above, this secures that the locally finite posets extracted by Sorkin's algorithm from the $\vec{\mathcal{U}}_i$s (which are now causally interpreted) can be viewed as (causal) simplicial complexes[164] and, *in extenso*, the incidence algebras (qausets) associated with them can be viewed as graded discrete differential algebras (manifolds) [92, 93, 128, 130] thus allowing the entire ADG-theoretic panoply to be applied on (the finsheaves of) them [74, 75], and

   — ii) as noted before, our operational scheme is in glaring contrast to Sorkin *et al.*'s more 'realistic' conception of dynamical (local) causality (gravity). For example, we recall from [115] that for Sorkin, in contradistinction to the rather standard operationalist or 'instrumentalist' interpretation of general relativity according to which the gravitational potentials, as represented by the ten components of the metric tensor $g_{\mu\nu}$, provide "*a summary of the behaviour of idealized clocks and measuring rods*" [38, 48, 112, 122], the gravitational field—the dynamical field of 'locality' or 'local causality' [74, 93]—"*is an independent substance, whose interaction with our instruments gives rise to clock-readings, etc*". This alone justifies the realist or 'Platonic' (ontological) causet hypothesis according to which "*spacetime, at small scales, <u>is</u> a locally finite poset*" [18]—a realm quite detached from and independent of (the operationalist or 'pragmatist' [45] philosophy according to which *all that there is and matters is*) "*what we actually do to produce spacetime by our measurements*" [115]—whose partial order is the discrete analogue of the relation that distinguishes past and future events in the (undoubtedly realistic or 'Platonic') macroscopic, geometrical spacetime continuum of general relativity.

8. We now come to the ADG-theoretic assumption of 'arithmetizing' or 'coordinatizing' our coarse localizations or measurements. This is represented by assuming that the base topological space X, which we have charted by covering it by the open gauges U in $\mathcal{U}_i$ (or equivalently, in $\vec{\mathcal{U}}_i$), is $\mathbf{K}$-algebraized in the sense that we localize sheaf-theoretically over it abelian $\mathbb{K} = \mathbb{R}, \mathbb{C}$-algebras which comprise the structure sheaf $\mathbf{A}_X$. The latter is supposed to be the commutative algebra sheaf of 'generalized arithmetics' in our theory—the

[163] Again, the right-pointing arrow over the covering $\mathcal{U}_i$ indicates the causal semantics '*coarse causal regions*' given to the open sets U in it above.

[164] It must be noted here that it was Finkelstein who first insisted, in a reticular and algebraic setting not very different from ours called 'the causal net', for *a causal version of (algebraic) topology and its associated (co)homology theory* [43].

realm in which our coarse local measurements, represented by the local sections of $\mathbf{A}$ (in $\Gamma(U, \mathbf{A}) \equiv \mathbf{A}(U),\ U \in \mathcal{U}_i$), take values—the readings on our abstract gauges so to speak. That we choose the stalks of $\mathbf{A}$ to be inhabited by *abelian* algebras is in accord with Bohr's quantum theoretic imperative according to which our measurements always yield commutative, c-numbers.[165] Furthermore, as it was also emphasized in the previous subsection, since the constant sheaf $\mathbf{K} = \mathbf{R}, \mathbf{C}$ of the reals or the complexes is canonically injected into $\mathbf{A}$, we realize again that *the usual numerical continua* $\mathbb{R}$ *and* $\mathbb{C}$ *enter into our theory via the process of abstract coordinatization and local measurement, and not by assuming that the base topological space(time)* X *is a classical, locally Euclidean continuum (ie, a manifold).* Finally, we must also emphasize here, as it was noted throughout the previous sections, that *in ADG all our (local) calculations reduce to expressions involving (local sections of)* $\mathbf{A}$*—in particular, all our vector sheaves* $\mathcal{E}$ *of rank* n *are (locally) of the form* $\mathbf{A}^n$[166] *and, as a result, their (local) structure symmetries comprise the matrix group* $(\mathcal{E}nd\mathcal{E}(U))^\bullet \equiv M_n(\mathbf{A}(U))^\bullet$.

9. Finally, anticipating our comments on an abstract, essentially categorical, version of gauge invariance and covariance of the gravitational dynamics of qausets in terms of finsheaf morphisms to be given subsequently, we note here that, although our kinematical, operational-algebraic conception of finitary quantum causality above is apparently observation or gauge $\mathcal{U}_i$-dependent [74], the dynamics, which is expressed in terms of the principal (fin)sheaf morphism—the finitary gravitational spin-Lorentzian connection $\vec{\mathcal{D}}_i$ and its scalar curvature $\vec{\mathcal{R}}(\vec{\mathcal{D}}_i)$, will be seen to be manifestly $\mathcal{U}_i$-independent. Thus, while quantum causality is kinematically expressed as a power relationship between events relative to our own coarse observations (gauges) of them in $\mathcal{U}_i$, its dynamical law of motion is characteristically independent of the latter [74]. We will comment further on this apparent paradox in 5.1.1.

4.3.1 Projective limits of fcqv-moduli spaces

In closing the present section, we would like to make some final kinematical remarks. These concern inverse limits of moduli spaces $\vec{\mathcal{M}}_i^{(+)}(\vec{\mathcal{E}}_i^\uparrow)$ of (self-dual) fcqv-spin-Lorentzian connections (dynamos) $\vec{\mathcal{D}}_i^{(+)}$ on the Lorentzian finsheaves $\vec{\mathcal{E}}_i^\uparrow := (\vec{\mathcal{E}}_i, \vec{\rho}_i)$. These spaces are defined as follows

$$\vec{\mathcal{M}}_i^{(+)}(\vec{\mathcal{E}}_i^\uparrow) := \vec{\mathcal{A}}_i^{(+)}(\vec{\mathcal{E}}_i^\uparrow)/\overrightarrow{\mathcal{A}ut}_i\vec{\mathcal{E}}_i^\uparrow \tag{120}$$

and they are the fcq-analogues of the ADG-theoretic moduli spaces defined in (92) in general, as well as in (103) and (104) in the particular case of self-dual connections.[167] $\vec{\mathcal{M}}_i^{(+)}(\vec{\mathcal{E}}_i^\uparrow)$, as we shall see in the next section, plays the role of the quantum configuration space for our theory which regards the (self-dual) fcqv-dynamos $\vec{\mathcal{D}}_i^{(+)}$ as (the sole) fundamental (quantum) dynamical variables.

[165] See also footnote 41 and [74, 75].

[166] And rather fittingly, the *local (coordinate) gauge* $e^U \equiv \{U;\ (e_i)_{0\leq i\leq n-1}\}$ ($U \in \mathcal{U}_i$) of the vector sheaf $\mathcal{E}$ of rank n in footnote 19, which consists of local sections of $\mathcal{E}$ (in $\mathcal{E}(U) \equiv (\mathbf{A}(U))^n \equiv \mathbf{A}^n(U)$), can be equivalently called *a local frame of* $\mathcal{E}$ [67].

[167] In (120), $\vec{\mathcal{A}}_i^{(+)}(\vec{\mathcal{E}}_i^\uparrow)$ is the fcq- (and self-dual) version of the abstract affine space $\mathsf{A}_{\mathbf{A}}(\mathcal{E})$ of $\mathbf{A}$-connections $\mathcal{D}$ on a vector sheaf $\mathcal{E}$ in (54).

Now, one such moduli space corresponds to (*ie*, is based on) each and every member of the direct system $\overset{\rightleftarrows}{\mathcal{T}}= \{\mathfrak{T}_i\}$ of fcq-differential triads and, *in extenso*, to each member of the inverse system $\overleftarrow{\mathcal{G}} = \{(\vec{\mathcal{P}}_i^{\uparrow}, \vec{\mathcal{D}}_i^{(+)})\}$ of principal Lorentzian finsheaves of qausets and their reticular (self-dual) spin-Lorentzian connections.[168] Thus, we can similarly define the projective system $\overleftarrow{\mathcal{M}} := \{\vec{\mathcal{M}}_i^{(+)}(\vec{\mathcal{E}}_i^{\uparrow})\}$ of (self-dual) fcqv-moduli spaces like the one in (120) and, according to the general ADG theory [81, 82], take its categorical limit, which yields

$$\mathcal{M}_{\infty}^{(+)}(\mathcal{E}_{\infty}^{\uparrow}) = \lim_{\infty \leftarrow i} \overleftarrow{\mathcal{M}} \equiv \lim_{\infty \leftarrow i}\{\vec{\mathcal{M}}_i^{(+)}(\vec{\mathcal{E}}_i^{\uparrow})\} \tag{121}$$

the $\mathcal{C}^{\infty}$-`smooth` moduli space of $\mathcal{C}^{\infty}(X)$-automorphism equivalent `smooth` (self-dual) spin-Lorentzian connections ${}^{(\mathbb{K})}\mathcal{D}_{\infty}^{(+)}$ on the Lorentzian vector bundle/sheaf $\mathcal{E}_{\infty}^{\uparrow}$ associated to the principal orthochronous Lorentzian bundle/sheaf ${}^{(\mathbb{K})}\mathcal{P}^{\uparrow} \equiv \vec{\mathcal{P}}_{\infty}^{\uparrow}$ over the region X of the $\mathcal{C}^{\infty}$-`smooth` $\mathbb{K}$-manifold M. As noted before, $\mathcal{M}_{\infty}^{(+)}(\mathcal{E}_{\infty}^{\uparrow})$ corresponds to a generalized version (*ie*, a $\mathcal{C}^{\infty}$-`smooth` one) of the classical moduli space $\mathcal{A}_{\infty}^{(+)}$ of gauge-equivalent (self-dual) $\mathcal{C}^{\infty}$-smooth spin-Lorentzian connections on the region X of the usual differential (*ie*, $\mathcal{C}^{\infty}$-smooth) spacetime $\mathbb{K}$-manifold M.

5 Locally Finite, Causal and Quantal Vacuum Einstein Equations

This is the neuralgic section of the present paper. Surprisingly, it is also the simplest one as it is essentially a straightforward transcription of the ADG constructions and results of sections 2 and 3 to the locally finite case of curved finsheaves of qausets $\vec{\mathcal{E}}_i^{\uparrow}$ and their reticular spin-Lorentzian connections $\vec{\mathcal{D}}_i$. So, without further ado, we are going to present a locally finite, causal and quantal version of the vacuum Einstein equations (53) for Lorentzian gravity emphasizing in particular their physical interpretation. We also derive these equations from an action principle.

5.1 Finitary, Causal and Quantal Vacuum Einstein-Lorentzian Gravity

First we note that the $\vec{\mathbf{A}}_i$-connection $\vec{\mathcal{D}}_i$ on $\vec{\mathcal{E}}_i^{\uparrow}$ is assumed to be compatible with the finsheaf morphism $\vec{\rho}_i$ in (114), as follows

$$\vec{\mathcal{D}}^i_{\mathcal{H}om_{\vec{\mathbf{A}}_i}(\vec{\mathcal{E}}_i^{\uparrow}, \vec{\mathcal{E}}_i^{\uparrow *})}(\vec{\rho}_i) = 0 \tag{122}$$

which is the finitary analogue of (17) implying that the connection $\vec{\mathcal{D}}_i$ is torsionless.[169] $\vec{\mathcal{D}}_i$ is a

[168] In fact, as we shall present in 5.5.2, a tower of numerous important inverse/direct systems of structures can be based on $\overset{\rightleftarrows}{\mathcal{T}}$. This just shows the importance of the notion of differential triad in ADG and its finitary application here.

[169] Note that in (122), to avoid subscript congestion on $\vec{\mathcal{D}}$, we have raised the refinement or finitarity index 'i' to a superscript.

reticular Lorentzian *metric* connection.

Then, analogously to the abstract expressions (36) and (37), and for the corresponding first prolongation $\vec{\mathcal{D}}_i^1$ of $\vec{\mathcal{D}}_i(\equiv \vec{\mathcal{D}}_i^0)$ as in (33) (*ie*, $\vec{\mathcal{D}}_i^1 : \vec{\mathbf{\Omega}}_i^1 \longrightarrow \vec{\mathbf{\Omega}}_i^2$), we define the non-zero curvature $\vec{R}_i$ of the reticular connection $\vec{\mathcal{D}}_i$ on $\vec{\mathcal{E}}_i^{\uparrow}$ as the following $\mathcal{E}nd\vec{\mathcal{E}}_i^{\uparrow}$-valued reticular 2-form

$$\begin{gathered}\vec{R}_i(\vec{\mathcal{D}}_i) := \vec{\mathcal{D}}_i^1 \circ \vec{\mathcal{D}}_i \neq 0 \\ \vec{R}_i \in \mathrm{Hom}_{\vec{\mathbf{A}}_i}(\vec{\mathcal{E}}_i^{\uparrow}, \vec{\mathbf{\Omega}}^2) = \mathcal{H}om_{\vec{\mathbf{A}}_i}(\vec{\mathcal{E}}^{\uparrow}, \vec{\mathbf{\Omega}}_i^2)(\vec{P}_i) = \vec{\mathbf{\Omega}}^2(\mathcal{E}nd\vec{\mathcal{E}}_i^{\uparrow})(\vec{P}_i)\end{gathered} \tag{123}$$

emphasizing also that it is an $\vec{\mathbf{A}}_i$-morphism. Thus, we can also define the associated Ricci tensor $\vec{\mathcal{R}}_i \in \mathcal{E}nd\vec{\mathcal{E}}_i^{\uparrow}$ as in (51) and the traced Ricci tensor corresponding to the reticular $\vec{\mathbf{A}}_i$-valued Ricci scalar curvature $\vec{\mathfrak{R}}_i$ as in (52).[170]

So, we are now in a position to write, at least formally, the locally finite, causal and quantal version of the vacuum Einstein equations for Lorentzian gravity (53), as follows

$$\vec{\mathfrak{R}}_i(\vec{\mathcal{E}}_i^{\uparrow}) = 0 \tag{124}$$

coining the pair $(\vec{\mathcal{E}}_i, \vec{\mathcal{D}}_i)$ consisting of a curved finsheaf of qausets $\vec{\mathcal{E}}_i$ and the non-trivial fcqv-dynamo[171] $\vec{\mathcal{D}}_i$ on it effecting that curvature, a *(f)initary, (c)ausal and (q)uantal (v)acuum Einstein field* (fcqv-E-field) and, *in extenso*, the triplet $(\vec{\mathcal{E}}_i, \vec{\rho}_i, \vec{\mathcal{D}}_i) \equiv (\vec{\mathcal{E}}^{\uparrow}, \vec{\mathcal{D}}_i)$ an *fcqv Einstein-Lorentz field* (fcqv-E-L-field). In turn, the latter prompts us to call the corresponding pentad $(\vec{\mathbf{A}}_i, \vec{\partial}_i \equiv \vec{d}_i^0, \vec{\mathbf{\Omega}}_i^1, \vec{\mathrm{d}}_i \equiv \vec{d}_i^1, \vec{\mathbf{\Omega}}_i^2)$ an *fcqv-E-L-curvature space*, which, in turn, makes the base causet $\vec{P}_i$ a *fcqv-E-space*.

5.1.1 Various interpretational matters

Now that we have formulated the vacuum Einstein equations for Lorentzian gravity on $\vec{\mathcal{E}}_i^{\uparrow} \equiv (\vec{\mathcal{E}}_i, \vec{\rho}_i)$, we wish to comment briefly on their physical meaning and other related issues of interpretation.

1. **Differentiability is independent of $\mathcal{C}^\infty$-smoothness.**[172] First we note, in keeping with our comments about 'reticular differential geometry' in part 4 of 4.1, that (124) is not a 'discrete differential' (*eg*, a difference) equation. Rather, it is a genuine, albeit abstract, differential equation. The discreteness of the base causet $\vec{P}_i$—the fcqv-E-space—does not prevent us from formulating genuine differential equations over it. As noted repeatedly earlier, $\vec{P}_i$ is merely a localization base (topological) for the qausets (living in the stalks of $\vec{\mathcal{E}}_i^{\uparrow}$) playing no role at all in the differential geometric structure of our theory. In other words, our differentials (*viz.*, connections) do not derive from the background space(time).

[170] Of course, we assume that, locally in the finsheaves, $\vec{\mathcal{R}}_i$ is a 0-cocycle of $n \times n$-matrices having for entries local sections of $\vec{\mathbf{\Omega}}_i^2$—that is to say, local 2-forms on $\vec{P}_i$, similarly to (38).

[171] See footnotes 100 and 101. We note here that one can straightforwardly write (124) in terms of a *self-dual* finitary spin-Lorentzian connection $\vec{\mathcal{D}}_i^+$ and its Ricci curvature scalar $\vec{\mathfrak{R}}_i^+$. We will return to self-dual connections in 5.3 where we will discuss a possible 'fully covariant' quantization scheme for vacuum Einstein Lorentzian gravity.

[172] This is the concluding slogan 2 in [75]. We will elaborate further on it in the last section.

Space(time) does not dictate to us the character of the differential mechanism, as we would be (mis)led to believe if we based ourselves on the classical differential geometry according to which differentiability comes from the $\mathcal{C}^\infty$-smooth manifold M or equivalently, from the coordinate algebras $\mathcal{C}^\infty(M)$ thereof. That our base space is 'discrete' does not mean at all that the differential geometric mechanism should also be so.

2. **A categorical dynamics and an abstract (generalized) principle of general covariance independent of** $\mathrm{Diff}(M)$**.** Related to 1, and as it was anticipated in [74], the dynamics of local quantum causality, as depicted in (124), is expressed solely in terms of (fin)*sheaf morphisms*—the main finsheaf morphism being the $\mathbf{C}$-linear fcqv-dynamo $\vec{\mathcal{D}}_i$. In fact, the fcqv-E-equations involve the curvature $\vec{\mathcal{R}}_i$ of the connection $\vec{\mathcal{D}}_i$, which moreover is an $\vec{\mathbf{A}}_i$-sheaf morphism. In other words, and in view of the physical interpretation that ADG gives to the commutative algebra sheaf $\mathbf{A}$ of generalized coefficients,[173] *the law for the fcqv-E-gravity is independent both of our (local) 'measurements' or 'geometry'* (as encoded in the structure sheaf of coefficients $\vec{\mathbf{A}}_i(V)$) *and of our (local) gauges* (represented by the open sets U in the open covering $\mathcal{U}_i$ that we employ to coarsely localize the events of X and 'measure' them in $\vec{\mathbf{A}}_i(V)$; V open in $\vec{P}_i$). This is reflected in the (local) gauge invariance of (124) under (local) transformations in $\overrightarrow{\mathcal{A}ut_i}\vec{\mathcal{E}}_i^{\uparrow}(V) \simeq M_n^i(\vec{\mathbf{A}}_i(V))^\bullet)$—the reticular (local) structure (gauge) group of $\vec{\mathcal{E}}_i^{\uparrow}(V) \simeq \vec{\mathbb{A}}_i^n(V)$. This invariance, in turn, is a consequence of the fact that both $\vec{\mathfrak{R}}_i$ and its contraction $\vec{\mathcal{R}}_i$ are gauge-covariant as they obey a reticular analogue of the homogeneous gauge transformation law for the gauge field strengths (39). Thus, as it has been already highlighted in [74], our scheme supports the following abstract categorical version of the principle of general covariance of general relativity:[174]

> *The fcqv-dynamics, as expressed in* (124), *is gauge $\mathcal{U}_i$-independent.* Accordingly, *the underlying topological spacetime X and its causal discretization $\vec{P}_i$ based on the locally finite open cover $\mathcal{U}_i$ play no role in the dynamics of local quantum causality* [74].

It is reasonable to expect this since the fcqv-dynamo $\vec{\mathcal{D}}_i$, or equivalently its fcqv-potential $\vec{\mathcal{A}}_i$, can be viewed as the 'generator' of the fcqv-dynamics.[175] and, as we argued in 1 above,

[173]See discussion around footnote 41.

[174]The epithet 'categorical' pertaining precisely to that both $\vec{\mathcal{D}}_i$ and $\vec{\mathcal{R}}_i(\vec{\mathcal{D}}_i)$ are morphisms ($\mathbf{K}$- and $\vec{\mathbf{A}}_i$-morphisms, respectively) in the relevant category of finsheaves of incidence algebras (qausets) over locally finite posets (causets).

[175]In the sense that the curvature $\vec{\mathcal{R}}_i(\vec{\mathcal{D}}_i)$—the dynamical variable in (106)—may be regarded as the 'measurable, geometric effect' since it is an $\vec{\mathbf{A}}_i$-morphism (*ie*, it respects our measurements), while $\vec{\mathcal{D}}_i$, from which $\vec{\mathcal{R}}_i$ derives and which is not an $\vec{\mathbf{A}}_i$-morphism (*ie*, it eludes our measurements!), as its 'original, algebraic cause'. That is why we called $\vec{\mathcal{D}}_i$ the fcqv-dynamo in the first place: it is the generator of the fcqv-dynamics (106)—the operator in terms of which the fcqv-E-equations are formulated. Subsequently, we will see how $\vec{\mathcal{D}}_i$ can be regarded as the main quantum configuration variable and $\vec{\mathcal{A}}_i$, the affine space of all such fcqv-dynamos, the corresponding kinematical space of quantum configurations (of $\vec{\mathcal{D}}_i$) in our theory.

differentiability is independent of the background causal-topological space $\vec{P}_i$[176] Thus, *a fortiori*

> *the fcqv-dynamics, as expressed in* (124), *is gauge $\mathcal{U}_i$-independent.* Accordingly, *the underlying topological spacetime X and its causal discretization $\vec{P}_i$ based on the locally finite open cover $\mathcal{U}_i$*[177] *play no role in the dynamics of local quantum causality as encoded in the fcqv-dynamo $\vec{\mathcal{D}}_i$ or in its potential $\vec{\mathcal{A}}_i$* [74].

Plainly then, the reticular invariance (gauge) group of (the vacuum dynamics of qausets (124) generated by $\vec{\mathcal{D}}_i$ on) $\vec{\mathcal{E}}_i^{\uparrow}$—the structure group $\overrightarrow{\mathcal{A}ut_i\mathcal{E}}_i^{\uparrow}$—has no relation whatsoever with the invariance group $\mathrm{Diff}(M)$ of the classical differential spacetime manifold M of general relativity. For instance, $\mathrm{Diff}(M)$, which implements the principle of general covariance in Einstein's classical theory of gravity, is precisely the group that preserves the differential (*ie*, $\mathcal{C}^\infty$-smooth) structure of the underlying spacetime manifold. In contradistinction, $\overrightarrow{\mathcal{A}ut_i\mathcal{E}}_i^{\uparrow}$, which locally is isomorphic to $M_n^i(\vec{\mathbf{A}}_i(U))^{\bullet}$,[178] is the group that preserves the local incidence algebraic structure of qausets stalk-wise in their finsheaf $\vec{\mathcal{E}}_i^{\uparrow}$ thus *it has nothing to do with the underlying topological base causet $\vec{P}_i$ per se.*[179] Of course, since, as we argued earlier, differentiability in ADG, and in our finitary theory in particular, derives from the stalk (*ie*, from the incidence algebras modelling qausets), the (local) gauge group $\overrightarrow{\mathcal{A}ut_i\mathcal{E}}_i^{\uparrow}$ of incidence algebra automorphisms, *like its classical analogue* $\mathrm{Diff}(M)$, *respects the reticular differential structure*, *but unlike* $\mathrm{Diff}(M)$, *it (and the reticular differential structure that it respects) does not come from the background causal-topological space $\vec{P}_i$.* All in all,

> *Dynamics in our ADG-based theory, as expressed in* (124), *is genuinely background spacetime-free*, whether the latter is a smooth continuum, or a locally finite causal space like a causet, or pretty much whatever else.

3. **Everything comes from dynamics: no a priori spacetime.** The last remarks in section 2 and the ones above bring to mind Einstein's philosophical remark

   "*Time and space are modes by which we think, not conditions in which we live*" [36].

   as well as Antonio Machado's insightful poetic verse

   "*Traveller there are no paths; paths are made by walking*" [63].

   in the sense that our theory (and ADG in general) indicates that spacetime is not something 'physically real'—*ie*, it is not an active substance that participates in the dynamics of

[176] The connection $\vec{\mathcal{D}}_i$ being in effect a generalized differential operator (derivation) of an essentially algebraic character [67, 68, 74, 75].

[177] See subsection 4.3 above.

[178] And $M_{n=4}^i(\vec{\mathbf{A}}_i(U))^{\bullet} \simeq sl(2,\mathbb{C})_i \simeq so(1,3)_i^{\uparrow}$ [74].

[179] In other words, $\overrightarrow{\mathcal{A}ut_i\mathcal{E}}_i^{\uparrow}$ acts directly on the (local) objects that live on 'space(time)' (*ie*, on the local sections of $\vec{\mathcal{E}}_i^{\uparrow}$—the qausets), not on 'space(time)' itself.

Nature. *The only physically significant entity in our theory is the dynamical fcqv-E-field* $(\vec{\mathcal{E}}_i^{\uparrow}, \vec{\mathcal{D}}_i)$,[180] which does not depend at all on a supporting space(time) (of any sort, 'discrete' or 'continuous') for its dynamical subsistence and propagation. This is in glaring contrast to the classical theory (general relativity) where spacetime is fixed *a priori*,[181] once and forever so to speak, by the theorist[182] to a background $\mathcal{C}^\infty$-smooth arena and it does not get involved into the dynamics[183] (*ie*, in the Einstein equations).

However, Machado's insight seems to go a bit further, for it intuits not only that space(time) is (physically) non-existent (because it is dynamically non-participatory), but also that it is actually the 'result' of dynamics.[184] How can we understand this in the context of ADG and what we have said so far? To give a preliminary answer to this question, we may have to address it first from a kinematical and then from a deeper dynamical perspective.

- **i) Spacetime from 'algebraic kinematics'.** The kinematical emergence of 'space' from incidence algebras modelling discrete quantum topological spaces and of 'space*time*' from the same structures, but when the locally finite partial orders from which they come from are interpreted *à la* Sorkin [115] as causal sets rather than as finitary topological spaces, has been worked out in [92, 93]. Especially in the second reference, the kinematics of a reticular, dynamically variable quantum spacetime topology—a Wheelerian foam-like structure so to speak—was worked out entirely algebraically based on a variant of Gel'fand duality[185] coined *Gel'fand spatialization*. The latter pertains to an extraction of *points* and the concomitant assignment of a suitable *topology* on them, by exploiting the structure and representation theory of (finite dimensional) non-abelian associative algebras like our incidence Rota algebras $\vec{\Omega}_i$ modelling qausets. Such a procedure, quite standard in algebraic geometry [107], is essentially based on first identifying points with *kernels of irreducible representations of the* $\vec{\Omega}_i$s which, in turn, are *primitive ideals* in the $\vec{\Omega}_i$s, and then endowing the collection of these ideals—the so-called *primitive spectra of the incidence algebras Spec*$(\vec{\Omega}_i)$—with a non-trivial topology.[186] Subsequently in [74], we heuristically argued that the very definition of the principal finsheaves $\vec{\mathcal{P}}_i^{\uparrow}$ of qausets over Sorkin *et al.*'s causets, which are interpreted as the kinematical structures of a locally finite, causal and quantal theoresis of Lorentzian spacetime and vacuum Einstein gravity, is essentially *schematic*.[187] The general lesson we have learned from this work is that

[180] In subsections 5.3 and 5.4 this remark will prove to be of great import since we will argue that our theory is 'fully covariant' and, in a subtle sense that we will explain, 'innately quantum'.

[181] That is to say, *there are paths!*

[182] That is to say, "time and space are modes by which *we* think..."—our own theoretical constructs or figments.

[183] That is to say, spacetime is not an active, dynamical, 'living' so to speak, condition.

[184] That is, "*paths are made by walking*".

[185] The reader will have to wait until the following subsection for more comments on Gel'fand duality.

[186] For the incidence algebras in focus such a topology is the *Rota topology* [92, 86, 93].

[187] In (noncommutative) algebraic geometry, schemes—a particular kind of 'ringed spaces'—are sheaves of (noncommutative) rings or algebras over their prime spectra usually endowed with the standard Zariski topology [107]. Incidentally, in ADG, the pair $(X, \mathbf{A})$, which has been coined '$\mathbf{K}$-algebraized space', maybe thought of as such (commutatively) ringed space [67]. The schematic aspects of our theory and their affinity to similar noncommu-

'space(time)' and its geometry[188] *is secondary, derivative from a deeper, purely algebraic theoresis of Physis*, *inherent* already *in the initial*, so to say thus far *'geometrical' aspect.*[189]

- **ii) Spacetime from 'algebraic dynamics'.** The idea that spacetime and gravity come from an algebraically modelled (quantum) dynamics is a deeper one than **i)**. Presumably, in Machado's verse quoted above,

  > *it is exactly the particles, fields and their mutual interrelations (ie, interactions) that 'do the walking', and by their dynamics they 'define' (ie, delimit) 'spacetime'.*[190]

  It must be noted that, still at the kinematical level of description, Euclidean geometry is an abstraction from the motions of, as well as the congruence and incidence relations between, rigid bodies. However, Einstein was the first to realize that geometry should not be regarded as an entity fixed *ab initio* by the theoretician, but it must be made part of the general physical process thus be subjected to dynamical changes [33], hence he arrived at general relativity the dynamical theory of the spacetime metric $g_{\mu\nu}$ [38]. On the other hand, very early on Einstein also realized that even though general relativity relativized the spacetime metric and successfully described it as a dynamical variable, the smooth geometric spacetime continuum was still lying at the background as an inert, non-dynamical, ether-like substance *a priori* fixed by the theorist [32, 34]; consequently, and intrigued by the dramatic paradigm-shift in physical theory that quantum mechanics brought about, he intuited soon after the formulation of general relativity that

  > "*...The problem seems to me how one can formulate statements about a discontinuum without calling upon a continuum space-time as an aid; the latter should be banned from theory as a supplementary construction not justified by the essence of the problem—a construction which corresponds to nothing real. But we still lack the mathematical structure unfortunately...*" (1916)[191]

tative, quantal topological spaces known as *quantales*, as well as to sheaves over such quantales (and the topoi thereof), have been explored in [88] and recently reviewed in [89, 90].

[188] We use the term 'geometry' in a general sense which includes for instance 'topology' and other qualities of 'space'.

[189] We tacitly abide to the broad 'definition' of geometry as '*the analysis of algebraic structure*'. It must also be noted here that Finkelstein has long maintained in a spirit akin to ours that *spacetime, causality, gauge fields and gravity* are *emergent notions from a more basic, purely algebraic (and finitistic!) theory* [42, 43, 103, 104, 105, 45, 106]; hence, *innately 'quantal'*.

[190] From this perspective, the standard procedure of first laying down the kinematics of a theory (*eg*, the space of kinematical histories or paths of the system) and then the dynamics, appears to be upside down. Dynamics ('cause') comes first, the kinematical space ('effect') second. This already points to a significant departure of our scheme from Sorkin *et al.*'s causet theory whose development followed Taketani and Sakata's methodological paradigm for the construction of a physical theory according to which *one must first develop (and understand!) the kinematics of a physical theory and then proceed to formulate the dynamics* [115]. Perhaps this is the way *we* have so far practiced and understood physics—*ie*, by first delimiting what can possibly happen (kinematics) and then describing what actually happens (dynamics)—but Physis herself may not work that way after all [73].

[191] This quotation of Einstein can be found in [119].

and a year before his death, that

> "*...An algebraic theory of physics is affected with just the inverted advantages and weaknesses* [than a continuum theory][192]*...It would be especially difficult to derive something like a spatio-temporal quasi-order from such a schema...But I hold it entirely possible that the development will lead there...*[that is,] *against a continuum with its infinitely many degrees of freedom.*" (1954)[193]

Also, again motivated by the quantum paradigm, he intuited that

> "*...Perhaps the success of the Heisenberg method points to a purely algebraic method of description of nature, that is to the elimination of continuous functions from physics.*" (1936) [35]

and, in the concluding remarks in the last appendix of *The Meaning of Relativity*, that

> "...[Quantum phenomena do] *not seem to be in accordance with a continuum theory, and must lead to an attempt to find a purely algebraic theory for the description of reality.*" (1956) [38]

In our theory, which rests on the intrinsically algebraic sheaf-theoretic axiomatics of ADG [67, 68], spacetime as such, especially in its classical $\mathcal{C}^\infty$-smooth guise, plays no operative role in the formulation of the fcqv-E-dynamics (124). All that is of mathematical import and physical significance in our scheme is the fcqv-E-field $(\vec{\mathbf{\Omega}}_i, \vec{\mathcal{D}}_i)$ the connection part of which—the fcqv-dynamo $\vec{\mathcal{D}}_i$—being of purely categorico-algebraic character. All that is physically meaningful in our model is $(\vec{\mathbf{\Omega}}_i \equiv \vec{\mathcal{E}}_i^{\uparrow}, \vec{\mathcal{D}}_i)$ and the dynamics (124) which it obeys. Furthermore, the quanta of the fcqv-E-field, which have been called *causons* in [74, 75], represent the dynamical 'elementary particles' of the (gauge) fcqv-potential field $\vec{\mathcal{A}}_i$ of quantum causality,[194] and by their algebraico-categorical dynamics they *define* the quantum gravitational vacuum without being dependent in any sense on an ambient spacetime—a background stage that just passively supports their dynamics.[195] At the same time, one may think of $\overrightarrow{\mathcal{A}ut_i}\vec{\mathcal{E}}_i^{\uparrow}$—the structure group of $\vec{\mathbf{\Omega}}_i$ where the reticular connection 1-form $\vec{\mathcal{A}}_i$ takes values—as the algebraic self-transmutations of the causon defining some sort of '*quantum causal foam*' [93].[196] Thus, we seem to find ourselves in accord with the quotation of Feynman in the previous section, since

[192] In square brackets and non-emphasized are our own completions of the text in order to enhance continuity and facilitate understanding.

[193] This quotation of Einstein can be found in [119].

[194] The reader should wait until our remarks on geometric (pre)quantization in subsection 5.4 where we make more explicit this 'fields$\longleftrightarrow$particles (quanta)' correspondence.

[195] We argued earlier that the role the base topological causet—the fcqv-E-space $\vec{P}_i$—plays in our theory is merely an auxiliary one: $\vec{P}_i$ is a substrate or 'scaffolding' that avails itself only for the sheaf-theoretic localizations of the dynamically variable qausets; nothing else.

[196] In a Kleinian sense, the geometry of the causon—the quantum of the algebraic fcqv-dynamo $\vec{\mathcal{D}}_i$ representing dynamical changes of (local) quantum causality in (the stalks of, *ie*, the sections of) $\vec{\mathcal{E}}_i^{\uparrow} \equiv \vec{\mathbf{\Omega}}_i$—is encoded in the (structure) group $\overrightarrow{\mathcal{A}ut_i}\vec{\mathbf{\Omega}}_i$ of its incidence algebraic automorphisms.

we actually avoid defining up-front the physical meaning of quantum geometry, fluctuating topology, space-time foam, etc., and instead we give geometrical meaning after quantization (algebraization).[197] *In broad terms, algebra precedes geometry, since the (algebraic dynamics of the) quantum precedes (geometrical) 'space'.*

In a similar vain, we note that, in the context of ADG, the fundamental difference noted at the end of subsection 2.3 between the notion of connection $\mathcal{D}$—a purely algebraic notion since, for instance, $\mathcal{A}$ transforms affinely (inhomogeneously) under the gauge group,[198] and its curvature $R(\mathcal{D})$—a purely geometric notion since it transforms tensorially under the automorphism group of the vector sheaf,[199] becomes very relevant here. For example, in connection with (124), we note that $\vec{\mathfrak{D}}_i$ may be viewed as the generalized algebraic differential operator in terms of which one sets up the fcqv-E-equations, while its curvature $\vec{\mathcal{R}}_i(\vec{\mathfrak{D}}_i)$ as the geometry (*ie*, the solution 'space') of those equations. Loosely speaking,

$\mathcal{D}$ stands to $R(\mathcal{D})$ as the 'cause' (algebra/dynamics) stands to the 'effect' (geometry/kinematics).[200]

Indeed, in [75], and based on the abstract version of the Chern-Weil theorem and the associated Chern isomorphism *à la* ADG, we similarly argued that the purely algebraic notion of connection $\mathcal{D}$ lies on the quantal side of the quantum divide (Heisenberg Scnhitt), while its geometric, 'observable' (*ie*, measurable) consequence—the curvature $R(\mathcal{D})$—on the classical side.[201] Moreover, in [75], based on general geometric pre-quantization arguments [67, 68, 69], we saw how the algebraic causon—the quantum of the connection $\vec{\mathfrak{D}}_i$—eludes our measurements, so that what we always measure is its field strength $\vec{\mathfrak{R}}(\vec{\mathfrak{D}}_i)$, never the connection itself. In a Bohrian sense, the classical, geometrical (because **A**-respecting) field strength is the result of our measuring the quantum (because **A**-eluding), algebraic connection.

In closing **ii)**, we would like to mention, also in connection with **i)** above, that even string theory, which purports to derive the classical spacetime manifold and Einstein's equations from a deeper quantum string dynamics, has recently focused on defining (spacetime) points and on deriving a topology for them by entirely algebraico-categorical means not very different, at least in spirit, from ours [9].

- **iii) No topology and no metric on 'space': an apparent paradox from categorical dynamics.** We would like to mention briefly the following apparently para-

[197] This remark hints at *our maintaining that our theory is*, to a great extent, *already or innately quantum* (so that the usual formal procedure of quantization of a classical theory, like general relativity, in order to arrive at a quantum theory of gravity—regarded as 'quantum general relativity—is 'begging the question' when viewed from the ADG-based perspective of our theory). After subsections 5.3 and 5.4, this claim of ours will become more transparent.

[198] That is to say, it does not respect our local measurements of (*ie*, the geometry of) the causon in $\vec{\mathbf{A}}_i(U)$.

[199] That is to say, it respects our local measurements of the causon in $\vec{\mathbf{A}}_i(U)$.

[200] See footnote 175.

[201] Revisit footnote 41.

doxical feature of our theory which has already been mentioned and resolved in [74]. While we started by covering the spacetime region X by the 'coarse' open gauges U in $\mathcal{U}_i$ thus we associated with the latter the base causal-topological space $\vec{P}_i$ and interpreted them as coarse observations or 'rough chartings' of the causal relations between events in X [74], at the end, that is to say, at the dynamical level, the dynamics of qausets over $\vec{P}_i$ is gauge $\mathcal{U}_i$-independent since it is expressed categorically in terms of the finsheaf morphisms $\vec{\mathcal{D}}_i$.[202] Thus, in the end the background space(time) seems to 'disappear' from the physical processes in the quantum deep as it plays no role in the gauge invariant dynamics of qausets. That this is only apparently and not really paradoxical has been explained in detail at the end of [74]. Here, and in connection with footnote 101, we would like to bring to the attention of the reader that the finitarity index (the degree of localization of our qausets) 'i' in (124) should not be mistaken as indicating that $\vec{\mathcal{D}}_i$ or its curvature $\vec{\mathcal{R}}_i$ are intimately dependent on the gauge $\mathcal{U}_i$, for, as we repeatedly argued before, *they are not*.[203] The index merely indicates that our structures are discrete and that (124) is the finitary analogue of the ADG-theoretic expression (53).[204] The corresponding statement that *the localization index is physically insignificant* is precisely what it was meant in [92, 74, 93] when we said that *the incidence algebras*, whether they are taken to model discrete quantum topological spaces proper [92, 93] or their causal analogues—qausets [86, 74], *are alocal structures* (*ie*, they are not vitally dependent on any pre-existent or *a priori* postulated and physically significant space(time)).

Now that we have shown both that the causal topology of the base causet $\vec{P}_i$ plays no role in the dynamics of qausets (124) and that differentiability comes from the incidence algebras in the stalks of the curved $\vec{\mathbf{\Omega}}_i$s, we are also in a position to return to footnote 17, the comparison between $\mathcal{D}$ and $R(\mathcal{D})$ in 2.4, as well as to our comments on the metric $\vec{\rho}_i$ in 'about the stalk' in 4.1, and note that in our algebraic connection-based (*ie*, gauge-theoretic) scenario

> *fcqv-E-L-gravity does not describe the dynamics of a vacuum spacetime metric as such in the way the original theory (ie, general relativity) does. Like the generalized differential $\vec{\mathcal{D}}_i$, the $\vec{\mathbf{A}}_i$ metric $\vec{\rho}_i$ is a finsheaf morphism, thus it is about the local (stalk-wise) algebraic structure of the gauged qausets, not about the underlying causal-topological $\vec{P}_i$ per se. Hence, on the face of* (124), *we agree with Feynman's hunch in* 3.1 *that "the fact that a massless spin-*2 *field can be interpreted as a metric was simply a coincidence that might be understood as representing some kind of gauge invariance".*

Of course, it is again plain that the finitarity index on the reticular metric $\vec{\rho}_i$ is of no

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[202]Equivalently, the curvature finsheaf morphism $\vec{\mathcal{R}}_i(\vec{\mathcal{D}}_i)$ in (124) is gauge $\mathcal{U}_i$-covariant.

[203]Quite on the contrary, as we said, since they are finsheaf morphisms, they show that they are $\mathcal{U}_i$-independent entities!

[204]As it were, the finitarity index shows that our theory is a concrete application of ADG to the locally finite regime of qausets; it is of no other physical significance.

physical (dynamical) significance since it, like the geometrical notion of curvature, is an $\vec{\mathbf{A}}_i$-respecting finsheaf morphism. Thus, $\vec{\rho}_i$, like $\vec{\mathcal{R}}_i$, lies on the classical (geometrical) side of the quantum divide.[205]

5.2 Derivation of fcqv-E-L Gravity from an Action Principle

We wish to emulate the situation in the abstract theory and derive (124) from the variation of a reticular, causal and quantal version $\overrightarrow{\mathfrak{E}\mathfrak{H}}_i$ of the E-H action functional $\mathfrak{E}\mathfrak{H}$. In the same way that the latter is a functional on the affine space $\mathsf{A}_{\mathbf{A}}^{(+)}(\mathcal{E}^{\uparrow})$ of (self-dual) Lorentzian $\mathbf{A}$-connections $\mathcal{D}$ on $\mathcal{E}^{\uparrow}$ taking values in the space $\mathbf{A}(X)$ of global sections of $\mathbf{A}$ (65), $\overrightarrow{\mathfrak{E}\mathfrak{H}}_i$ is a functional on the space $\vec{\mathcal{A}}_i^{(+)}(\vec{\mathcal{E}}_i^{\uparrow})$[206] of the (self-dual) fcqv-E-L-dynamos $\vec{\mathcal{D}}_i^{(+)}$ on $\vec{\mathbf{\Omega}}_i$ taking values in $\vec{\mathbf{A}}_i(\vec{P}_i)$, as follows

$$\overrightarrow{\mathfrak{E}\mathfrak{H}}_i : \ \vec{\mathcal{A}}_i^{(+)}(\vec{\mathbf{\Omega}}_i) \longrightarrow \vec{\mathbf{A}}_i(\vec{P}_i) \tag{125}$$

reading 'point-wise' in $\vec{\mathcal{A}}_i^{(+)}(\vec{\mathbf{\Omega}}_i)$

$$\vec{\mathcal{A}}_i^{(+)}(\vec{\mathbf{\Omega}}_i) \ni \vec{\mathcal{D}}_i^{(+)} \mapsto \overrightarrow{\mathfrak{E}\mathfrak{H}}_i(\vec{\mathcal{D}}_i) := \vec{\mathcal{R}}_i^{(+)}(\vec{\mathcal{D}}_i^{(+)}) := tr\vec{\mathcal{R}}_i^{(+)}(\vec{\mathcal{D}}_i^{(+)}) \tag{126}$$

where, plainly, $\vec{\mathcal{R}}_i^{(+)}$ is a global section of the structure finsheaf $\vec{\mathbf{A}}_i$ of reticular coefficients over the base causet $\vec{P}_i$ (*ie*, $\vec{\mathcal{R}}_i^{(+)} \in \vec{\mathbf{A}}_i(\vec{P}_i)$).[207]

At this point we recall the basic argument from 3.3: to be able to derive (124) from the variation (extremization) of $\overrightarrow{\mathfrak{E}\mathfrak{H}}_i$ with respect to $\vec{\mathcal{D}}_i \in \vec{\mathcal{A}}_i(\vec{\mathbf{\Omega}}_i)$, all we have to secure is that the derivative $\overbrace{\overrightarrow{\mathfrak{E}\mathfrak{H}}_i(\vec{\mathcal{D}}_i\gamma(t))}^{\cdot} |_{t=0}$, for a path $\gamma(t)$ in the reticular spin-Lorentzian connection space $\vec{\mathcal{A}}_i(\vec{\mathcal{E}}_i^{\uparrow})$ ($\gamma : \ \mathbb{R} \longrightarrow \vec{\mathcal{A}}_i(\vec{\mathbf{\Omega}}_i)$), is well defined. The latter means in turn that there should be a well defined notion of convergence, limit and, of course, a suitable topology on the structure sheaf $\vec{\mathbf{A}}_i$ relative to which these two notions make sense.

We recall from [92, 86, 87, 93, 74] that the abelian (structure) subalgebras $\vec{\mathbf{A}}_i$ of the incidence algebras $\vec{\Omega}_i$ modelling the qausets in the stalks of the $\vec{\mathbf{\Omega}}_i$s can then be construed as carrying a (natural) topology—the so-called *Rota topology*—provided by the $\vec{\Omega}_i$s' structure (primitive ideal) spaces (Gel'fand duality).[208] With respect to the (now quantum causally interpreted) Rota

[205] As it should, since it is *us*—the observers—that carry on local acts of measurement on 'it' (*ie*, the quantum system 'spacetime') and obtain *c*-numbers in the process all of which are effectively encoded in ρ. Indeed, geometry (and measurement) without a metric sounds as absurd as convergence (and continuity) without a topology.

[206] We write $\vec{\mathcal{A}}_i^{(+)}$ for $\vec{\mathcal{A}}_{\vec{\mathbf{A}}_i}^{(+)}$. We met earlier $\vec{\mathcal{A}}_i^{(+)}$ in connection with the definition of the reticular moduli spaces $\vec{\mathcal{M}}_i^{(+)}(\vec{\mathcal{E}}_i^{\uparrow})$ in (120).

[207] In what follows, we will forget for a while the epithet 'self-dual' (and the corresponding notation) for the gravitational connection and its curvature. We will return to self-dual $\vec{\mathcal{D}}_i$s a bit later.

[208] In the next subsection we will comment further on the rich import that Gel'fand duality has in our theory.

topology, it has been shown that there is a well defined notion of (discrete) convergence and, *in extenso*, of limits [114, 129, 20, 92, 86, 87, 93]. Thus, $\overbrace{\overrightarrow{\mathfrak{E}\mathfrak{H}}_i(\vec{\mathcal{D}}_i\gamma(t))}^{\cdot}|_{t=0}$ is well defined.

5.3 Towards a Possible Covariant Quantum Dynamics for the Finitary Spin-Lorentzian Connections

We have seen how general relativity can be cast as a Y-M-type of gauge theory in finitary terms, that is to say, how it may be expressed solely as the dynamics of a fcqv-spin-Lorentzian connection variable—the dynamo $\vec{\mathcal{D}}_i$. These dynamos have been already '*kinematically quantized*' [74] and '*geometrically (pre)-quantized*' to *causons* [75][209] along the lines of ADG [67, 68, 69].

In the present subsection we discuss the possibility of developing a covariant path integral-type of *quantum dynamics* for the finitary spin-Lorentzian dynamos $\vec{\mathcal{D}}_i$ on the respective $\vec{\mathcal{E}}_i^{\uparrow} \equiv \vec{\mathbf{\Omega}}_i$s. As a first step, we wish to emulate formally the usual practice in the quantum gauge theories of matter (*ie*, QED, QCD and higher dimensional Y-M theories of a semi-simple and compact Lie structure group $\mathcal{G}$) whereby a covariant quantum dynamics is represented by a path integral over the space of the relevant connections on the corresponding principal fiber bundles over a $\mathcal{C}^\infty$-smooth spacetime manifold M (a $\mathcal{G} = U(1)$-bundle for QED, a $\mathcal{G} = SU(3)$-bundle for QCD and $\mathcal{G} = SU(N)$-bundles for general Y-M theories). Thus, in our case too, we intuit that the main object of study should be the following 'heuristic device'

$$\vec{\mathcal{Z}}_i := \int_{\vec{\mathcal{A}}_i(\vec{\mathcal{E}}_i^{\uparrow})} e^{i\overrightarrow{\mathfrak{E}\mathfrak{H}}_i} d\vec{\mathcal{A}}_i \tag{127}$$

where $\vec{\mathcal{A}}_i(\vec{\mathcal{E}}_i^{\uparrow})$ is the affine space of finitary spin-Lorentzian connections $\vec{\mathcal{D}}_i$ on the curved orthochronous Lorentzian finsheaves $\vec{\mathcal{E}}_i^{\uparrow} \equiv \vec{\mathbf{\Omega}}_i$ of qausets which is thus being regarded as the (quantum) kinematical configuration space (of 'fcqv-dynamo or causon quantum histories') of our theory. More precisely, due to the local reticular gauge invariance of our theory, the actual physical configuration space is the fcqv-analogue $\vec{\mathcal{M}}_i(\vec{\mathcal{E}}_i^{\uparrow}) := \vec{\mathcal{A}}_i(\vec{\mathcal{E}}_i^{\uparrow})/\overrightarrow{\mathcal{A}ut}_i(\vec{\mathcal{E}}_i^{\uparrow})$ of the moduli space in (103) that we defined earlier in (120), and it consists of finitary gauge-equivalent fcqv-connections $\vec{\mathcal{D}}_i$. We thus recast (127) as follows

$$\vec{\mathcal{Z}}_i := \int_{\vec{\mathcal{M}}_i} e^{i\overrightarrow{\mathfrak{E}\mathfrak{H}}_i} d([\vec{\mathcal{A}}_i]_{\overrightarrow{\mathcal{A}ut}_i\vec{\mathcal{E}}_i^{\uparrow}}) \tag{128}$$

where $[\vec{\mathcal{A}}_i]_{\overrightarrow{\mathcal{A}ut}_i(\vec{\mathcal{E}}_i^{\uparrow})}$ denotes the gauge $\overrightarrow{\mathcal{A}ut}_i\vec{\mathcal{E}}_i^{\uparrow}$-equivalence classes of fcqv-gravitational connections $\vec{\mathcal{D}}_i$ on $\vec{\mathcal{E}}_i^{\uparrow}$—the elements of $\vec{\mathcal{M}}_i(\vec{\mathcal{E}}_i^{\uparrow})$.

In what follows we enumerate our anticipations and various remarks about $\vec{\mathcal{Z}}_i$ in (128) by gathering information from both the canonical (*ie*, Hamiltonian) approach to quantum general relativity and the covariant path integral (*ie*, Lagrangian or action-based) approach to

[209]With a concomitant sheaf-cohomological classification of the corresponding associated curved line sheaves $\mathcal{L}$ inhabited by these causons. We will return to make more comments on geometric (pre)quantization in subsection 5.4.2.

Lorentzian quantum gravity. In particular, and in connection with the former approach, we discuss issues arising from Ashtekar's self-dual connection variables scenario for both classical and quantum gravity [3] as well as from their $\mathcal{C}^\infty$-smooth loop holonomies—the so-called loop formulation of (canonical) quantum gravity [100][210]—especially viewed under the functional analytic (C^*-algebraic) prism of Ashtekar and Isham [6, 7]. We thus commence our exposition with a brief review of both the Hamiltonian (canonical) and the Lagrangian (covariant) approaches to Lorentzian quantum gravity.

5.3.1 The canonical (Hamiltonian) approach: Ashtekar variables

More than fifteen years ago, Ashtekar [3] proposed a new set of variables for both classical and quantum general relativity essentially based on a complex spacetime manifold and a self-dual connection version of the Palatini comoving 4-frame (*vierbein*) formulation of gravity. The main assumptions were the following:

- A 4-dimensional, complex, orientable, $\mathcal{C}^\infty$-smooth spacetime manifold M of Lorentzian signature.
- The basic gravitational variable $\mathcal{A}^+_\infty$,[211] which is a $so(1,3)_{\mathbb{C}}$-valued self-dual connection 1-form.
- The *vierbein* variable e, which defines a vector space isomorphism between the tangent space of M and a fixed 'internal space' $\mathcal{M}$ equipped with the usual Minkowski metric η and the completely antisymmetric tensor ϵ. $\mathcal{A}^+_\infty$ is self-dual with respect to ϵ.[212]

In the new variables $\mathcal{A}^+_\infty$ and e, the gravitational action functional assumes the following so-called *first-order form*

$$S_{ash}[\mathcal{A}^+_\infty, e] = \frac{1}{2}\int_M \epsilon(e \wedge e \wedge R^+_\infty) \tag{129}$$

[210] For reviews of the loop approach to quantum gravity and relevant references, the reader is referred to [62, 98].

[211] The index '∞' just indicates that $\mathcal{A}$ *is a* $\mathcal{C}^\infty$*-smooth connection on* M.

[212] More analytically and in bundle-theoretic terms (note: most of the items to be mentioned in this footnote should be compared one-by-one with the corresponding ADG-theoretic ones defined earlier and the reader must convince herself that, ADG-theoretically, we possess all the classical smooth vector bundle-theoretic notions and constructions *without any notion of* $\mathcal{C}^\infty$*-smoothness being used*. This observation will prove crucial in the sequel—see comparison between our ADG-based finitary scheme and the usual $\mathcal{C}^\infty$-approaches to non-perturbative canonical or covariant Lorentzian quantum gravity that the present footnote will trigger after (140)), one lets $\mathcal{T}$—equipped with a pseudo-Riemannian metric η and fixed orientation $\mathcal{O}$—be an 'internal Minkowskian bundle space' isomorphic to the tangent bundle TM. $\mathcal{O}$ and η define a nowhere vanishing global section ϵ of $\wedge^4\mathcal{T}^*$. The aforesaid fiber bundle isomorphism is symbolized as $e:\ TM \longrightarrow \mathcal{T}$, and its inverse e^{-1} is the comoving 4-frame field (*vierbein*) mentioned above (by pushing-forward e one can also define a volume form ϖ on M, while TM inherits via e^{-1} the metric η from $\mathcal{T}$). η similarly defines an isomorphism between $\mathcal{T}$ and its dual $\mathcal{T}^*$. Fortunately, in 4 dimensions, η and ϵ determine a unipotent Hodge-$\star$ operator: $\star:\ \wedge^2\mathcal{T} \longrightarrow \wedge^2\mathcal{T}$. One then regards as basic dynamical fields in Ashtekar's theory the aforementioned spin-Lorentzian metric (*ie*, η-preserving) connection 1-form $\mathcal{A}^+_\infty$ (whose curvature R^+_∞ is a section of $\wedge^2\mathcal{T}\otimes\wedge^2\mathcal{T}^*$ and satisfies relative to $\star$ the self-duality relation: $\star R^+_\infty = R^+_\infty$) and the frame field e (which is a $\mathcal{T}$-valued 1-form: $e \in \Omega^1(\mathcal{T})$). (Of course, one can also transfer via e^{-1} the connection $\mathcal{A}^+_\infty$ from $\mathcal{T}$ to TM.)

which may be readily compared with the usual Palatini action

$$S_{pal}[\mathcal{A}_\infty, e] = \frac{1}{2}\int_M \epsilon(e \wedge e \wedge R_\infty) \tag{130}$$

and directly see that S_{ash} is S_{pal}'s self-dual version.[213] We also note that, upon variation of both S_{ash} and S_{pal} with e, one obtains the vacuum Einstein equations (*ie*, that $\mathcal{A}_\infty^{(+)}$ is Ricci-flat), while upon variation with $\mathcal{A}_\infty^{(+)}$, one obtains the metric-compatibility condition for $\mathcal{A}_\infty^{(+)}$ (*ie*, that it is the gauge potential part of the Levi-Civita connection of the metric).

The attractive feature of Ashtekar's new variables is that in terms of them one can simplify and write neatly the Hamiltonian constraints for gravity, thus one obtains a clear picture of how to proceed and canonically quantize the theory *à la* Dirac. To revisit briefly the Hamiltonian approach, one assumes that M factors into two submanifolds: $M = \Sigma^3 \times \mathbb{R}$,[214] thus securing the $3+1$ decomposition needed in order to approach quantum gravity canonically. Then, one assumes as *configuration space* of the theory the affine space ${}^3\mathcal{A}_\infty^+$ of complex, smooth, self-dual, $so(3)_{\mathbb{C}}$-valued connections ${}^3\mathcal{A}_\infty^+$ on Σ^3,[215] and as *phase space* the cotangent bundle $T^*({}^3\mathcal{A}_\infty^+)$ coordinatized by canonically conjugate pairs $({}^3\mathcal{A}_\infty^+, {}^3E_\infty)$[216] obeying the following Poisson bracket relations[217]

$$\{{}^3\mathcal{A}_\infty^+, {}^3E_\infty\} = \delta^3(x-y);\ (x, y \in \Sigma^3) \tag{131}$$

In terms of these variables, the Hamiltonian for gravity can be shown to be [218]

$$H(\mathcal{A}, E) = \int_{\Sigma^3} (\frac{1}{2}\lambda_l \epsilon RE^2 + i\lambda_s RE)d^3x \tag{132}$$

with λ_l and λ_s being Lagrange multipliers corresponding to the well known lapse and shift functions in the canonical formulation of gravity.

On the other hand, since the theory has internal (gauge) $SO(3)_{\mathbb{C}}$-symmetries and external (spacetime) $\mathrm{Diff}(M)$-symmetries, not all points (classical states) in the phase space $T^*({}^3\mathcal{A}_\infty^+)$ can be regarded as being physical. This is tantamount to the existence of the following five first-class constraints for gravity[219]

$$\begin{array}{c} \text{one Gauss divergence constraint (internal)} : \mathcal{D}E = 0 \\ \text{three spatial diffeos constraints (external)} : RE = 0 \\ \text{one temporal diffeo constraint (external)} : \epsilon RE^2 = 0 \end{array} \tag{133}$$

[213] Plainly, $R_\infty^{(+)}$ in both (129) and (130) is the curvature of the (self-dual) connection $\mathcal{A}_\infty^{(+)}$.

[214] Assuming also that the 'spatial' or 'spacelike' 3-submanifold Σ^3 is orientable and compact.

[215] Thus, in this picture gravity may be thought of as an $SO(3)_{\mathbb{C}}$-gauge theory—the dynamical theory of ${}^3\mathcal{A}_\infty^+$ in the connection space ${}^3\mathcal{A}_\infty^+$. Shortly we will see that gravity is actually a 'larger' theory transformation-wise: *it is an $SO(3)_{\mathbb{C}}$-gauge theory together with* $\mathrm{Diff}(M)$*-constraints coming from assuming up-front that there is an external background $\mathcal{C}^\infty$-smooth spacetime manifold.*

[216] Where ${}^3E_\infty$ is a smooth vector density representing a generalized electric field on Σ^3.

[217] In (131), we present indexless symplectic relations. The reader is referred to [62] for the more elaborate indexed relations.

[218] Again, all indices, including the ones for $\mathcal{A}$ and E above, are omitted in (132).

[219] Again, all indices are suppressed for symbolic economy and clarity.

which must be satisfied by the (classical) physical states.[220] At the same time, $\mathcal{D}E$, RE and ϵRE^2 can be seen to generate local gauge transformations in the internal gauge space, as well as Σ^3-spatial diffeos and $\mathbb{R}$-temporal diffeos respectively in the external $M = \Sigma^3 \times \mathbb{R}$-spacetime manifold,[221] thus they transform between physically indistinguishable (equivalent) configurations. It is important to note here that pure Y-M theory also has the internal Gauss gauge constraint, but not the other four external 'spacetime diffeomorphism' $\mathrm{Diff}(M)$-constraints. Due to this fact, Loll points out for example that "*pure gravity may be interpreted as a Yang-Mills theory with gauge group* $\mathcal{G} = SO(3)_{\mathbb{C}}$, *subject to four additional constraints in each point of* Σ"[222] [62]. We will return to this remark soon. One should also notice here that since the integrand of $H(\mathcal{A}, E)$ in (131) is an expression involving precisely these four external spacetime $\mathrm{Diff}(M)$-constraints, the Hamiltonian vanishes on physical states.[223] Since, as noted in footnote 221, H is the generator of the smooth time-evolution of Σ^3 in the spacetime manifold M, one says (even at the classical level) that, at least from the canonical viewpoint, gravity is 'inherently' a no-time ('time-less') theory.

A straightforward canonical quantization of gravity *à la* Dirac would then proceed by the following standard formal replacement of the Poisson bracket relations in (131) by commutators

$$\{\mathcal{A}, E\} = \delta^3(x-y) \longrightarrow [\widehat{\mathcal{A}}, \widehat{E}] = i\delta^3(x-y);\ (x, y \in \Sigma^3) \tag{134}$$

with the hatted symbols standing now for field operators acting on the unphysical phase space $T^*({}^3\mathcal{A}^+_\infty)$[224] which is suitably 'Hilbertized'. The latter pertains essentially to the promotion of the space $\mathcal{F}({}^3\mathcal{A}^+_\infty) = \{\Psi(\mathcal{A})\}$ of $\mathbb{C}$-valued functions on ${}^3\mathcal{A}^+_\infty$ to a Hilbert space $\mathcal{H}$ of physical states. This is usually done in two steps:

- First, in order to take into account the gauge and diffeomerphism invariance of the theory, one projects out of $\mathcal{F}({}^3\mathcal{A}^+_\infty)$ all the wave functions lying in the kernel of the corresponding operator expressions of the gravitational constraints in (133). These are precisely the *physical* quantum states-to-be, as they satisfy operator versions of the constraints (*ie*, they are annihilated by them). They comprise the following subspace $\mathcal{F}_p$ of (p)hysical wave functions in $\mathcal{F}$

$$\mathcal{F}_p := \{\Psi(\mathcal{A}) :\ \widehat{\mathcal{D}E}\Psi(\mathcal{A}) = \widehat{RE}\Psi(\mathcal{A}) = \widehat{\epsilon RE^2}\Psi(\mathcal{A}) = 0\} \tag{135}$$

  where the hatted symbols denote operators.

[220] In (133) the temporal-diffeomorphisms constraint is commonly known as the *Hamiltonian constraint*.
[221] The Hamiltonian constraint generates the smooth time-evolution of Σ^3 in M.
[222] Which we call Σ^3 here.
[223] This is characteristic of gravity regarded as a gauge theory on a $\mathcal{C}^\infty$-smooth spacetime manifold M, namely, $\mathrm{Diff}(M)$, which implements the principle of general covariance, is (part of) gravity's gauge (structure) group $\mathcal{G}$.
[224] As can be read from [62] for instance, there are (technical) reasons for using $T^*({}^3\mathcal{A}^+_\infty)$ instead of the physical cotangent bundle $T^*({}^3\mathcal{A}^+_\infty/\mathcal{G})$ on the 3-connection moduli space ${}^3\mathcal{A}^+_\infty/\mathcal{G}$. We will comment on some of them subsequently when we will emphasize the need to develop a differential geometry on the moduli space of gauge-equivalent connections.

- Then, one promotes $\mathcal{F}_p$ to a Hilbert space $\mathcal{H}_p$ by endowing it with the following hermitian inner product structure

$$< \Psi_2(\mathcal{A})|\Psi_1(\mathcal{A}) >:= \int_{{}^3\mathcal{A}^+_\infty/\mathcal{G}} \Psi_2^*(\mathcal{A})\Psi_1(\mathcal{A})[d\mathcal{A}]_{\mathcal{G}} \tag{136}$$

thus essentially by insisting that the wave functions $\Psi(\mathcal{A})$ are square-integrable with respect to $< . | . >$. However,

> *so far one has not been able to find a fully $\mathcal{G}$-invariant (ie, $SO(3)_{\mathbb{C}}$-gauge and* $\mathrm{Diff}(M)$*-invariant) integration measure $[d\mathcal{A}]_{\mathcal{G}}$ on ${}^3\mathcal{A}^+_\infty/\mathcal{G}$.*[225]

This is essentially the content of the so-called *inner product problem* in the canonical approach to quantum general relativity.

Before we move on to discuss briefly the covariant path integral approach to quantum gravity, which, as we shall see, also encounters a similar 'diffeomorphism-invariant measure over the moduli space of connections' problem, we wish to present some elements from the Ashtekar-Isham analysis of the loop approach to canonical quantum gravity [100, 6, 7]. Of particular interest to us, without going into any technical detail, are two general features of this analysis: (i) the application of a version of Gel'fand duality on the space of Y-M and (self-dual) gravitational connections in a spirit not so different from how we use Gel'fand duality in our algebraico-sheaf-theoretic approach to causal sets here, and, as a result of this application, (ii) its pointing to a generalized integration theory over the moduli space ${}^3\mathcal{A}^+_\infty/\mathcal{G}$ in order to deal with the '$\mathrm{Diff}(M)$-invariant measure problem' mentioned in connection with the Hilbert space inner product in (136).

In [100], Rovelli and Smolin used non-local, gauge-invariant Wilson loops—the traces of holonomies of connections around closed loops in Σ^3[226]—and found physical states for canonical quantum gravity, that is to say, ones that are annihilated by the aforementioned operator constraints. Remarkably enough, they found that such states can be expressed in terms of knot and link-invariants (which themselves are $\mathbb{C}$-valued functions on knots and links that are invariant under spatial diffeos), thus they opened new paths for exploring the apparently intimate relations that exist between gauge theories, (quantum) gravity, knot theory and, *in extenso*, the geometry

[225] Of course, the tough problem is finding a $\mathrm{Diff}(M)$-invariant measure, not an $SO(3)_{\mathbb{C}}$ one. Ingenious ideas, involving abstract or generalized integration theory, have been used in order to actually construct such a $\mathrm{Diff}(M)$-invariant measure [11, 12]. We will return shortly to comment a bit more on abstract integration theory and generalized measures. Also, motivated by this remark about $\mathrm{Diff}(M)$-invariant measures, from now on we will abuse notation and identify the gauge (structure) group $\mathcal{G}$ of gravity only with its external smooth spacetime manifold symmetries (*ie*, $\mathcal{G} \equiv \mathrm{Diff}(M)$) and forget about its internal, 'purely gauge', $SO(3)_{\mathbb{C}}$-invariances.

[226] One defines a Wilson loop as follows: $W^\rho_{\mathcal{A}^{(+)}}(\ell) := tr \exp_{po}(\oint_{\ell\in\Sigma^3} \mathcal{A}^{(+)})$, where ℓ is a spatial loop (in Σ^3), ρ is a (finite dimensional, complex) matrix representation of the Lie algebra $\mathbf{g}$ of the gauge group $\mathcal{G}$ where the (self-dual) connection $\mathcal{A}^{(+)}$ takes values (in our case, $so(3)_{\mathbb{C}}$, and the index '*po*' to exp denotes '*path ordered*' [62]. For the sake of completeness, we note that Rovelli and Smolin, based on Ashtekar's new variables $(\mathcal{A}, e)$, actually defined an 'adjoint' set of Wilson loop variables that reads: $W^\rho_e(\ell) := tr[e(\ell) \exp_{po}(\oint_{\ell\in\Sigma^3} \mathcal{A}^{(+)})]$.

of low-dimensional manifolds.[227] Such promising new research possibilities aside, what we would like to highlight here are certain features in the aforesaid work of Ashtekar and Isham which put Rovelli and Smolin's loop variables on a firm and rigorous mathematical footing, and, in particular, opened the way towards finding $\mathcal{G}$-invariant measures (as well as generalized integrals to go with them) that could help us resolve problems like the one of the inner product mentioned above.

Our first remark concerns the general moduli space $\mathcal{A}_\infty/\mathcal{G}$ of gauge theories and gravity. We have seen above what a crucial role it plays both in the classical and the quantum descriptions of these theories. For one thing, it is the classical configuration space of the theories in their connection-based formulation. As we have said, to get the classical phase space, one deals with the cotangent bundle $T^*(\mathcal{A}_\infty/\mathcal{G})$.[228] In their quantum versions, the moduli space $\mathcal{A}_\infty/\mathcal{G}$ is supposed to give way to the Hilbert space $L^2(\mathcal{A}_\infty/\mathcal{G}, d\mu)$ of $\mathbb{C}$-valued, square-integrable functions on $\mathcal{A}_\infty/\mathcal{G}$ with respect to some measure $d\mu$, which is in turn expected to be $\mathcal{G}$-invariant. However, due to $\mathcal{A}_\infty/\mathcal{G}$'s infinite dimensionality, non-linear nature and rather 'complicated' topology,[229] there are significant (technical) obstacles in finding (*ie*, actually constructing!) such a $d\mu$. Moreover, in the canonical approach, the loop variables of Rovelli and Smolin provide us with a set of manifestly $\mathcal{G}$-invariant configuration observables, but we lack analogous gauge-invariant momentum observables not least because the differential geometry of the moduli space $\mathcal{A}_\infty/\mathcal{G}$ (and *in extenso* of the cotangent bundle $T^*(\mathcal{A}_\infty/\mathcal{G})$) has not been well developed or understood.[230] These are some of the technical difficulties one encounters in trying to develop classical $\mathcal{C}^\infty$-smooth differential geometric ideas on spaces of gauge-equivalent connections and exactly because of them one could 'justify' the ADG-theoretic perspective we have adopted in the present paper.[231]

Now, what Ashtekar and Isham did in order to deal with some of the problems mentioned in the previous paragraph is to 'downplay' the structure of the space $\mathcal{A}_\infty/\mathcal{G}$ *per se* and rather *work directly with the functions that live on that space.*[232] Thus, they defined the so-called *holonomy C^*-algebra* $\mathfrak{C} = \mathcal{F}(\mathcal{A}_\infty/\mathcal{G})$ of $\mathbb{C}$-valued functions on $\mathcal{A}_\infty/\mathcal{G}$ generated by Wilson loops $W(\ell)$ like the ones mentioned in footnote 226.[233] $\mathfrak{C}$ was straightforwardly seen to be abelian, thus by using the well known Gel'fand-Naimark representation theorem they identified $\mathfrak{C}$ with the

227 Refer to [14] for a thorough exposition of the close interplay and the fertile exchange of ideas between knot theory, gauge theory and (quantum) gravity.

228 The elements of $T^*(\mathcal{A}_\infty/\mathcal{G})$ are the classical physical observables of the theories.

229 This refers to the usual $\mathcal{C}^\infty$ (Schwartz) topology [65].

230 Principally motivated by this ellipsis, and as we noted earlier, [8] explores further the possibility of developing classical (*ie*, $\mathcal{C}^\infty$-smooth) differential geometry on $\mathcal{A}_\infty/\mathcal{G}$.

231 The reader is referred to [71] for a more elaborate ADG-theoretic treatment of moduli spaces of connections *vis-à-vis* gauge theories and gravity.

232 This is well in line with the general philosophy of ADG which we have repeatedly emphasized throughout this paper and according to which, in order to gather more information and gain more insight about (the structure of) 'space'—whatever that may be—one should look for an 'appropriate' algebra that encodes that information in its very structure. Then, in order to recover 'space' and perform the ever-so-useful in physics calculations (*ie*, 'geometrize' or 'arithmetize' the abstract algebraic theory so to speak), one should look for suitable representations of this algebra.

233 It must be noted however that *real* connections $\mathcal{A}$ were employed in [6]. The reader should not be concerned about this technical detail here.

commutative C^*-algebra $\overline{\mathfrak{F}}$ of continuous $\mathbb{C}$-valued functions on a compact Hausdorff topological space $\mathfrak{M} \equiv Spec(\mathfrak{C})$—the so-called *Gel'fand spectrum of* $\mathfrak{C}$.[234] In turn, every (continuous and cyclic) representation $\overline{\mathfrak{F}}$ of $\mathfrak{C} \equiv \mathfrak{F}$ has $L^2(\mathrm{Max}(\mathfrak{C}))$ as carrier Hilbert space with respect to some regular measure $d\mu$ on $\mathfrak{M}$ and, plainly, the representatives of the $\mathbb{C}$-valued Wilson loop operators in $\mathfrak{C}$ act on the elements Ψ of $L^2(\mathfrak{M})$ by multiplication.

Thus, while $\mathcal{A}_\infty/\mathcal{G}$ is the classical configuration space, quantum states Ψ naturally live on $\mathrm{Max}(\mathfrak{C})$ and can be thought of as 'generalized' gauge-equivalent connections. In fact, Rovelli and Smolin conceived of a deep correspondence between the spaces of (functions on) gauge-equivalent connections and (functions on) loops, which could be mathematically implemented by the following heuristic integral 'device'

$$\mathfrak{T}[\Psi(\ell)] := \int_{\mathcal{A}_\infty/\mathcal{G}} tr(\exp_{po} \oint_\ell \mathcal{A})\Psi([\mathcal{A}]_\mathcal{G})d\mu([\mathcal{A}]_\mathcal{G}) \tag{137}$$

called the (non-linear and in general non-invertible[235]) *loop transform*—a variant of the usual functional-analytic Gel'fand transform.[236] Again, in $\mathfrak{T}(\Psi)$ we witness the need to find measures on $\mathcal{A}_\infty/\mathcal{G}$.[237]

This last remark brings us to the main point we wish to make about the importance of the (abelian) C^*-algebraic point of view (and the application of the Gel'fand spectral theory that goes with it) on the moduli space of connections adopted by Ashtekar and Isham based on the Rovelli-Smolin loop representation of Ashtekar's new variables in the context of canonical quantum general relativity:

> the holonomy C^*-algebraic perspective on $\mathcal{A}_\infty/\mathcal{G}$ makes it clear that *one must adopt a 'generalized integration theory'*[238] in order to cope with integrals such as (136) and (137) and with the measures involved in them.

The idea to use 'generalized' or 'abstract measures' becomes 'natural' in Ashtekar and Isham's work as follows: as we noted above, the holonomy C^*-algebra $\mathfrak{C} = \mathfrak{F}(\mathcal{A}_\infty/\mathcal{G})$ is first transcribed

[234] The points of $Spec(\mathfrak{C})$ are kernels of (irreducible) representations of $\mathfrak{C}$ to $\mathbb{C}$ (*ie*, homomorphisms of $\mathfrak{C}$ to $\mathbb{C}$ commonly known as 'characters'), with the latter being the 'standard' abelian involutive algebra. In turn, these kernels are maximal ideals in $\mathfrak{C}$, so that equivalently one writes $\mathrm{Max}(\mathfrak{C})$ for $\mathfrak{M} \equiv Spec(\mathfrak{C})$ (in the sequel, we will use $Spec(\mathfrak{C})$, $\mathfrak{M}$ and $\mathrm{Max}(\mathfrak{C})$ interchangeably). $\mathrm{Max}(\mathfrak{C})$ carries the standard Gel'fand topology and the elements of $\overline{\mathfrak{F}}$ are continuous with respect to it. (Memo: the Gel'fand topology on $\mathfrak{M}$ is the weakest (coarsest) topology with respect to which the functions in $\overline{\mathfrak{F}}$ are continuous [65].)

[235] The loop transform is supposed to carry one from the connection to the loop picture, and back via $\mathfrak{T}^{-1}$. However, for $\mathfrak{T}^{-1}$ to exist, a set of (algebraic) constraints—the so-called Mandelstam constraints—must be satisfied by Wilson loops [6, 62].

[236] The Gel'fand transform may be viewed as a generalized Fourier transform [65]. The reader is encouraged to read from [6] a suggestive comparison made between the loop and the Fourier transform. For an ADG-theoretic use of the Gel'fand transform, in case $\mathbf{A}$ is a topological algebra sheaf (the 'canonical' example of a unital, commutative topological algebra being, of course, $\mathcal{C}^\infty(M)$—see remarks on Gel'fand duality in 5.5.1), the reader is referred to [67, 68].

[237] In (137), $[\mathcal{A}]_\mathcal{G}$ represents a class of $\mathcal{G}$-equivalent connections in $\mathcal{A}_\infty$—an element of the moduli space $\mathcal{A}_\infty/\mathcal{G}$.

[238] The reader should refer to [11, 12] for a relatively recent treatment of generalized $\mathrm{Diff}(M)$-invariant measures on moduli spaces of non-abelian Y-M and gravitational connections.

by the Gel'fand-Naimark representation to the C^*-algebra $\overline{\mathfrak{F}}$ of bounded, continuous, $\mathbb{C}$-valued functions on $\mathfrak{C}$'s spectrum $\mathrm{Max}(\mathfrak{C})$ having for carrier Hilbert space $L^2(\mathrm{Max}(\mathfrak{C}), d\mu)$. *How can we realize the measure $d\mu$ and the integral with respect to it*?

> The aforesaid idea of 'generalized measures' can be materialized in the C^*-algebraic context by identifying $\int[\cdot]d\mu$ with *a state σ on* $\overline{\mathfrak{F}}$—a (normalized, positive) linear form on $\overline{\mathfrak{F}}$, which is a member of $\overline{\mathfrak{F}}^*$. Then one thinks of $\sigma(f)$ as an abstract expression of $\int f\, d\mu$ ($f \in \overline{\mathfrak{F}}$). In turn, having this integral in hand, the inner product on $L^2(\mathrm{Max}(\mathfrak{C}))$ can be realized as $< \Psi_2|\Psi_1 >= \int \Psi_2^*\Psi_1 d\mu = s(\Psi_2^*\Psi_1)$.[239]

We now move on to discuss briefly the covariant path integral (Lagrangian) approach to quantum gravity, so that afterwards we can comment 'cumulatively' from an ADG-theoretic viewpoint on the heuristic integral $\vec{\mathcal{Z}}_i$ appearing in (128) in comparison with what we have said about both the canonical and the covariant quantization schemes for gravity.[240]

5.3.2 The covariant (Lagrangian) approach: the $\mathrm{Diff}(M)$-invariant path integral measure problem

One of the main disadvantages of any approach to the quantization of gravity based on the canonical formalism is the latter's breaking of full covariance by the unphysical $3+1$ space-time split that it mandates. In the Ashtekar approach for instance, one must choose a time-slicing by arbitrarily foliating spacetime into spacelike hypersurfaces on which the self-dual connection variables $\mathcal{A}^+_\infty$—the main dynamical variables of the theory—are defined and canonical Poisson bracket (classical) (131) or commutator (quantum) (134) relations are imposed.[241] The basic idea of a path integral quantization of gravity is not to force any such physically *ad hoc* $3+1$ split, thus retain full covariance of the theory.

In a Lagrangian, (self-dual) connection-based formulation of gravity in a $\mathcal{C}^\infty$-smooth spacetime manifold (like Ashtekar's in (129), but in all four spacetime dimensions), the path integral would be the following heuristic object

$$\mathcal{Z}_\infty = \int_{{}^4\mathcal{A}^{(+)}_\infty} e^{i[{}^4S^{(+)}_{ash}]}d\mathcal{A} \tag{138}$$

where the integral is taken now over all the (self-dual) $\mathcal{C}^\infty$-connections ${}^4\mathcal{A}^{(+)}_\infty$ over the whole 4-dimensional spacetime manifold M, and $S^{(+)}_{ash}$ is the 4-dimensional version of the Ashtekar

[239] With Ψ_2^* the complex conjugate of Ψ_2 (note: the reader should not confuse this $*$-star with the linear dual $*$-star in $\overline{\mathfrak{F}}^*$.

[240] Since both of these schemes are essentially based on the classical differential geometry of the $\mathcal{C}^\infty$-smooth spacetime manifold M (*ie*, they belong to category 1 in the prologue—in other words, *they are '$\mathcal{C}^\infty$-smoothness conservative'*) which ADG evades, such a comparison is relevant here and well worth the effort.

[241] Also, by such a $3+1$ decomposition one secures a well defined Cauchy problem for the dynamical equations (global hyperbolicity).

action (129) of the (self-dual) smooth connection variable ${}^4\mathcal{A}_\infty^{(+)}$.[242] Of course, again due to the $\mathcal{G} \equiv \mathrm{Diff}(M)$-invariance of the theory, one would expect the 'physical' path integral to be

$$\mathcal{Z}_\infty = \int_{{}^4\mathcal{A}_\infty^{(+)}/\mathcal{G}} e^{i[{}^4S_{ash}^{(+)}]} d([\mathcal{A}]_\mathcal{G}) \tag{139}$$

which, however, in order to make sense (even if only 'heuristically'!) care must be taken to make sure that one integrates over a single member ${}^4\mathcal{A}_\infty^{(+)}$ from each gauge equivalence class $[{}^4\mathcal{A}_\infty^{(+)}]_\mathcal{G}$ in ${}^4\mathcal{A}_\infty^{(+)}/\mathcal{G}$. Among the aforementioned problems of developing differential (and now integral) calculus on the moduli space of (non-abelian) gauge (Y-M) theories and gravity, is the fact that $\pi: \ \mathcal{A}_\infty \longrightarrow \mathcal{A}_\infty/\mathcal{G}$, regarded as a principal $\mathcal{G}$-bundle, is non-trivial, that is to say, it has no continuous global sections, which in turn means that there is no unique gauge choice, no unique fixing or selecting a single ${}^4\mathcal{A}_\infty^{(+)}$ from each $[{}^4\mathcal{A}_\infty^{(+)}]_\mathcal{G}$ in ${}^4\mathcal{A}_\infty^{(+)}/\mathcal{G}$. This is essentially the content of the well known Gribov ambiguity in the usual $\mathcal{C}^\infty$-fiber bundle-theoretic treatment of gauge theories.[243]

All in all, however, again it all boils down to finding a measure $d([\mathcal{A}]_\mathcal{G})$—in fact, a $\mathrm{Diff}(M)$-invariant one, since (139) involves smooth connections on a $\mathcal{C}^\infty$-spacetime manifold M—on the moduli space ${}^4\mathcal{A}_\infty^{(+)}/\mathcal{G}$. Thus, we see how both the non-perturbative canonical and the covariant approaches to quantum gravity, whose formulation vitally depends on the classical differential geometric apparatus provided by the $\mathcal{C}^\infty$-smooth manifold (in fact, by the structure coordinate ring $\mathcal{C}^\infty(M)$) and its structure group $\mathrm{Diff}(M)$, encounter the problem of finding a $\mathrm{Diff}(M)$-invariant measure on their respective moduli spaces. Below we argue how the ADG-theoretic basis, on which our finitary, causal and quantal vacuum Einstein gravity (124) and its possible covariant path integral quantization (128) rest, bypasses completely significant obstacles that these 'conventional' approaches[244] to quantum general relativity encounter. Altogether, we emphasize that our approach is genuinely background $\mathcal{C}^\infty$-smooth spacetime-free, fully covariant and that, based on the fact that arguably all diseases (*ie*, singularities, unrenormalizable infinities and other clas-

[242]However, it must be emphasized here that a $3+1$ space-time split is in a sense also implicit here. $\mathcal{Z}_\infty$ in (138) is normally regarded as a *transition amplitude* and the dynamical transition that it pertains to is between 'boundary spatial configuration 3-geometries'—say, $\Phi_1[{}^3\mathcal{A}_1^{(+)}]_{\Sigma_1^3}$ and $\Phi_2[{}^3\mathcal{A}_2^{(+)}]_{\Sigma_2^3}$—with the bulk 4-spacetime geometry interpolating between them. One usually writes $\mathcal{Z}_\infty|_{\Phi_1}^{\Phi_2} \equiv <\Phi_2|\Phi_1>= \int_{\Phi_1}^{\Phi_2} e^{i[{}^4S_{ash}^{(+)}]} d\mathcal{A}$.

[243]The reader should refer to [68] for a more elaborate, albeit formal, treatment, from an ADG-theoretic perspective, of the Gribov ambiguity *à la* Singer [111]. What must be briefly mentioned here is that the ADG-theoretic treatment of the Gribov ambiguity in [68] marks the commencement of the development of a full-fledged differential geometry—again of a non-classical, non-$\mathcal{C}^\infty$-smooth type—on the moduli space of gauge-equivalent connections. For instance, one could take as starting point for this development the following motivating question: *what is the structure of the 'tangent space' $T(\mathcal{O}_\mathcal{D}, \mathcal{D})$ to an orbit $\mathcal{O}_\mathcal{D}$ of a connection $\mathcal{D}$ in the affine space $A_\mathbf{A}(\mathcal{E})$ of* **A**-*connections on a vector sheaf $\mathcal{E}$?* For example, in 3.4 we saw that, ADG-theoretically, $T(\mathcal{O}_\mathcal{D}, \mathcal{D})$ can be identified with $\mathcal{S}_\mathcal{D}^\perp$ (98) and, as a result, $T(M(\mathcal{E}), \mathcal{O}_\mathcal{D})$ with $T(\mathcal{O}_\mathcal{D}, \mathcal{D})$'s orthogonal complement (*ie*, $\mathcal{S}_\mathcal{D}$!) (101). However, for the latest results from the most analytical ADG-theoretic treatment of moduli spaces of connections, the reader should refer to [71].

[244]'Conventional' here means 'classical', in the sense that all these approaches are based on the usual differential geometry of $\mathcal{C}^\infty$-manifolds. As we time and again said before, these are approaches that belong to the category 1 of 'general relativity and manifold conservative' scenarios mentioned in the prologue.

sical differential geometric anomalies) come from assuming up-front M, it is doubtful whether any '$\mathcal{C}^\infty$-conservative' attempt to quantize general relativity (by essentially retaining M) will be able to succeed.[245]

In connection with the last remarks, cogent arguments coming from [43, 44, 56] further support the position that the attempt to quantize gravity by directly quantizing general relativity (*ie*, by trying to quantize Einstein's equations in order to arrive at the quantum of the gravitational force field—the graviton) is futile, if one considers the following telling analogy:

> *it is as if one tries to arrive at the fine structure of the water molecule by quantizing the Navier-Stokes equations of hydrodynamics.*

We definitely agree with this position; however, as we saw before and we will crystallize in the next subsection, we would not go as far as to maintain that in order to arrive at a genuinely quantum theoresis of gravity one should first arrive at a quantum description of (the background) spacetime structure itself, for *spacetime does not exist* (*ie*, it has no physical meaning). Rather, going quite against the grain of theories that advocate either a 'continuous' (classical) or a 'discrete' (quantum) spacetime, we will hold that a genuinely covariant approach to quantum gravity should involve solely the dynamical fields (and their quanta) without any dependence on an external 'spacetime substrate', whether the latter is assumed to be 'discrete' or 'continuous'. This is what we mean by a '*fully covariant*' (and '*already quantum*') picture of gravity: *only the dynamical gravitational field (and its quanta), and no ambient (external/background) spacetime which forces one to consider its (ie, the spacetime's) quantization, exists.*

5.4 Cutting the Gordian Knot: No $\mathcal{C}^\infty$-Smooth Base Spacetime Manifold M, no $\mathrm{Diff}(M)$, No Inner Product Problem, no Problem of Time, a 'Fully Covariant', 'Purely Gauge-Theoretic' Lorentzian Quantum Gravity

In the present section we show how our finitary, ADG-based scheme for 'discrete' Lorentzian quantum gravity totally avoids three huge problems that the differential manifold M,[246], or more precisely, its 'structure group' $\mathcal{G} \equiv \mathrm{Diff}(M)$[247] presents to both the canonical and the covariant $\mathcal{C}^\infty$-manifold based approaches to quantum gravity.

First, we would like to state up-front the main lesson we have learned from ADG, which lesson, continuing the trend started in [75], we wish to promote to the following slogan:[248]

[245] Even more iconoclastically, in the following subsection we will maintain that *our scheme is already quantum*, so that *the quest for a quantization of gravity is in effect 'begging the question'*.

[246] Or ADG-theoretically, the assumption of $\mathcal{C}^\infty_M$ for structure sheaf $\mathbf{A}$.

[247] Here the term 'structure group' is not used exactly in the usual principal bundle and gauge-theoretic sense. Rather fittingly, it pertains to the 'symmetries' of the structure sheaf $\mathbf{A}$, which in the classical case is identified with $\mathcal{C}^\infty_M$.

[248] This is the second slogan in the present paper. Recall the first one from the beginning of section 4.

> **Slogan 2.** *One can do differential geometry without using any notion of calculus*; or what amounts to the same, *without using at all (background) differential (ie, $\mathcal{C}^\infty$-smooth) manifolds* [67, 68, 77, 78, 70, 69, 74, 72, 75, 73, 71].

Thus, in the present paper, where ADG was applied to the finitary-algebraic regime to formulate a causal and quantal version of vacuum Einstein Lorentzian gravity, *no classical differential geometric concept, construction, or result, and, of course, no background (or base) $\mathcal{C}^\infty$-smooth spacetime manifold, was used.* Precisely in this sense, our formulation of (124) and its covariant quantum version (128) *is genuinely background manifold-free or $\mathcal{C}^\infty$-smoothness independent.*

Another basic moral of ADG which is invaluable for its direct application to (quantum) gravity and (quantum) Y-M theories, and which nicely shows its manifest evasion of the classical differential geometry of $\mathcal{C}^\infty$-manifolds, can be expressed diagrammatically as follows

$$
\begin{tikzcd}
\boxed{\mathrm{CDG} \equiv \mathcal{C}^\infty-\mathrm{Manifolds}} \arrow[r, "(a)", "\mathbf{A}\equiv\mathcal{C}^\infty_X"'] & \boxed{\text{Tangent Bundles}} \arrow[r, "(b)", "\mathbf{A}\equiv\mathcal{C}^\infty_X"'] & \boxed{\mathcal{C}^\infty-\text{Vector Fields}} \arrow[d, "(c)"', "{\substack{(\mathcal{C}^\infty-\text{connections})\\ \mathcal{C}^\infty-\text{derivations}}}"] \\
\boxed{\mathrm{ADG} \equiv \text{arbitrary base } X} \arrow[u, "(c')", "\mathbf{A}\equiv\mathcal{C}^\infty_X\ (X\subset M)"'] \arrow[r, "\text{arbitrary } \mathbf{A}_X", "(a')"'] & \boxed{\text{Vector Sheaves}} \arrow[u, "(d')", "\mathbf{A}\equiv\mathcal{C}^\infty_X\ (X\subset M)"'] \arrow[r, "{\substack{(\mathbf{A}-\text{connections})\\ \text{sheaf morphisms}}}", "(b')"'] & \boxed{\text{Differential Equations}}
\end{tikzcd}
\tag{140}
$$

which we can put into words again in the form of a slogan:

> **Slogan 3.** Unlike the Classical Differential Geometry (CDG), whose (conceptual) development followed the path
>
> $$\mathrm{CDG} \equiv \text{Smooth Manifolds} \xrightarrow{(a)} \text{Tangent bundles} \xrightarrow{(b)}$$
> $$\text{Smooth Vector Fields} \xrightarrow{(c)} \text{Differential Equations}(\equiv \text{Physical Laws})$$
>
> schematically described in (140), and which can be read as follows: *the smooth manifold was made for the tangent bundle, which in turn was made for the vector fields, which were finally made for the differential equations (modelling the local laws of classical physics;*[249]*)* in contradistinction, the development of ADG followed the path
>
> $$\mathrm{ADG} \xrightarrow{(a')} \text{Vector Sheaves} \xrightarrow{(b')} \text{Differential Equations}$$
>
> which can be read as follows: *ADG refers in an algebraico-categorical way directly to the dynamical fields—represented by pairs such as '$(\mathcal{E}, \mathcal{D})$'—without the intervention (neither conceptually nor technically) of any notion of (background geometric manifold) space(time), or equivalently, independently of any intervening coordinates.* In other words, *ADG deals directly with the differential equations (the laws of physics),*

[249]In the concluding section we will return to comment further on the fact that the assumption of a differential manifold ensures precisely that the dynamical laws of physics obey the classical principle of locality.

which now are 'categorical equations' between sheaf morphisms—the $\mathbf{A}$*-connections* $\mathcal{D}$ *acting on the (local) sections of vector sheaves* $\mathcal{E}$ *under consideration.* Of course, one can recover CDG from ADG by identifying one's structure sheaf $\mathbf{A}$ with $\mathcal{C}^{\infty}_{M}$[250] thus, in effect, 'descend' from abstract, algebraic in nature, vector sheaves to the usual smooth vector or frame (tangent) bundles over (to) the geometrical base spacetime $\mathcal{C}^{\infty}$-manifold M ($c^{'}, d^{'}$).

5.4.1 Avoiding the problems of Diff(M) by avoiding M

Below, we mention three problems that our finitary-algebraic, ADG-based perspective on quantum gravity manages to evade completely. We choose to pronounce these problems via a comparison between the canonical and the covariant $\mathcal{C}^{\infty}$-manifold based approaches to quantum general relativity described above, and our ADG-theoretic locally finite, causal and quantal Lorentzian vacuum Einstein gravity. In particular, we initiate this comparison by basing our arguments on the contents of footnote 212 which makes it clear what the essential assumptions about the $\mathcal{C}^{\infty}$-approaches to quantum gravity are, and it also highlights their characteristic absence from our ADG-founded theory. In this way, the value of the slogans 1–3 above can be appreciated even more.

1. The fundamental assumption of all the non-perturbative $\mathcal{C}^{\infty}$-conservative approaches to quantum gravity, whether Hamiltonian or Lagrangian, is that *there is a background geometrical spacetime which is modelled after a* $\mathcal{C}^{\infty}$*-smooth base manifold* M. Thus, the point-events of M are coordinatized by $\mathcal{C}^{\infty}$-smooth functions whose germs generate the classical structure sheaf $\mathbf{A} \equiv \mathcal{C}^{\infty}_{M}$; hence, the natural 'structure group' of all those M-based scenarios is $\mathcal{G} \equiv \mathrm{Diff}(M)$.

2. The next assumption (of great import especially to the canonical approach via the Ashtekar variables) we can read directly from footnote 212: there is a (frame) bundle isomorphism e between TM and an 'internal' Minkowskian bundle $\mathcal{T}$,[251] whose inverse e^{-1} defines a local *vierbein* (4-frame) field variable on M[252] and secures the faithful transference of the classical $\mathcal{C}^{\infty}$-differential geometric structures, such as the smooth (self-dual) connections $\mathcal{A}^{(+)}_{\infty}$, the smooth Lorentzian metric η, the volume form ϖ, the smooth vector fields (derivations) and covectors (differential forms) *etc*, from $\mathcal{T}$ to TM.[253] In a nutshell, e^{-1} ensures that TM comes fully equipped with the classical (tangent bundle) differential geometric apparatus.

---

[250] (a) in (140).

[251] We may coin e *the (local) 'external' Lorentzian* $\mathcal{C}^{\infty}$*-manifold* M *soldering form.* It may be thought of as the 'umbilical cord' that ties (and feeds!) all the differential geometric constructions used in non-perturbative canonical or covariant quantum general relativity with (from) the background smooth manifold M.

[252] By abusing notation, we also denote the *vierbein* by e.

[253] Hence our calling e above a (local) 'external' Lorentzian $\mathcal{C}^{\infty}$-manifold M soldering form. (Recall also from footnote 212 that η, which is pulled back by e^{-1} from $\mathcal{T}$ to TM, effects the canonical isomorphism between TM—inhabited by vectors/derivations tangent to M, and its dual TM^{*}—inhabited by covectors/forms cotangent to M.)

3. When it comes to (especially the canonical) dynamics, one can easily see how this $\mathcal{C}^\infty$-spacetime bound language gives independent physical existence and 'reality' to the background (*ie*, 'external' to the dynamical fields themselves) geometrical smooth spacetime continuum itself, by statements such as,

   > "*In this approach*[254] *the action of diffeomorphism group gives rise to two constraints on initial data: the diffeomorphism constraint, which generates diffeomorphisms preserving the spacelike hypersurface, and the Hamiltonian constraint, which generates diffeomorphisms that move the surface in a timelike direction.*"[255]

   In the canonical Ashtekar approach, this is concisely encoded in the assumption that the smooth 4-frame field e is an independent (local) dynamical variable along with the (self-dual) smooth spin-Lorentzian connection 1-form $\mathcal{A}_\infty^{(+)}$.[256]

By striking contrast, our finitary, causal and quantal ADG-based approach to Lorentzian vacuum Einstein gravity assumes neither M[257] (and, as a result, no $\mathrm{Diff}(M)$ either), but perhaps more importantly, nor e. ADG in a sense cuts the 'umbilical cord' (e) that ties (and sustains differential geometrically) the $\mathcal{C}^\infty$-conservative approaches to (by) the background spacetime manifold M, and it concentrates directly on (the physical laws for) the dynamical objects—in our case, the (self-dual) fcqv-dynamos $\vec{\mathcal{D}}_i^{(+)}$—that live and propagate on 'it'. All in all we must emphasize that

> *the sole dynamical variable in our scheme is the reticular (self-dual) spin-Lorentzian connection variable* $\vec{\mathcal{D}}_i^{(+)}$ (in fact, *the fcqv-E-L-field* $(\vec{\mathcal{E}}_i^\uparrow, \vec{\mathcal{D}}_i)$) *and ADG enables us to formulate directly the dynamical equations for it without having to account for (ie, without the mediation and support of) a background geometrical smooth spacetime manifold* M. *In this sense, our ADG-theoretic, connection-based approach is more algebraic and more 'pure gauge-theoretic' (ie, 'fully covariant'—see below) than the approaches to gravity which are based on the classical* $\mathcal{C}^\infty$*-differential geometry of the smooth spacetime manifold (eg, Ashtekar's). At the same time, since there is no 'external' spacetime manifold, there is no need either to perform the necessary for the canonical quantization procedure* $3+1$ *space-time split which, as we contended earlier, breaks manifest covariance. Furthermore,* $\mathrm{Diff}(M)$ *is now replaced, in a Kleinian sense* [73], *by the structure group* $\overrightarrow{\mathcal{A}ut_i}$ *of* $\vec{\mathbf{A}}_i$*-automorphisms of* $\mathcal{E}_i^\uparrow$ *(ie, the group of the*

[254]That is, the canonical approach to quantum general relativity *à la* Ashtekar.

[255]Taken from the preface of the book "*Knots and Quantum Gravity*" where [7] and [62] belong. The constraints mentioned in this excerpt are precisely the four 'external' $\mathcal{C}^\infty$-smooth spacetime manifold $\mathrm{Diff}(M)$-constraints in (133).

[256]And recall from (129) and (130) that the vacuum Einstein equations are obtained from deriving the Palatini-Ashtekar action functionals with respect to e.

[257]Thus it gives the smooth spacetime manifold no independent physical (dynamical) reality 'external' to the dynamical gravitational gauge field itself (represented by the connection).

reticular transformations of the causon field itself—its dynamical self-transmutations so to speak[258]*). All in all, our approach is fully (gauge) covariant.*[259]

Now that we have stated, and analyzed in glaring contrast to the $\mathcal{C}^\infty$-conservative canonical and covariant approaches to quantum general relativity, the three slogans underlying our fcqv-approach to Lorentzian vacuum Einstein gravity, we are in a position to show how our theory simply evades the following three caustic issues for non-perturbative quantum gravity:

1. *The inner product problem*: in the canonical approach, this refers to the problem of fixing the inner product in the Hilbert space of physical states by requiring that it is invariant under $\mathrm{Diff}(M)$. As noted earlier, in effect it is the problem of finding a $\mathrm{Diff}(M)$-invariant measure. The same technical problem (*ie*, the problem of finding a $\mathrm{Diff}(M)$-invariant measure) essentially persists in the fully covariant path integral quantization approach to quantum general relativity (138) and (139). Since our theory is genuinely $\mathcal{C}^\infty$-smooth manifold M-free, thus also manifestly $\mathrm{Diff}(M)$-independent, it simply avoids the inner product problem. We thus write

$$\boxed{\text{No smooth manifold } M} \Rightarrow \boxed{\text{No Diff}(M)} \Rightarrow \boxed{\text{No inner product problem}} \tag{141}$$

However, it must be said that if one employs finite dimensional (Hilbert) space representations for the incidence algebras modelling qausets as in [129, 93][260] and one regards the latter spaces as inhabiting the stalks of associated finsheaves to the $\vec{\mathcal{P}}_i^{\uparrow}$s, or even if one just works with the aforesaid associated line sheaves $\vec{\mathcal{L}}_i$ of states of 'bare' or free causons, the issue of finding well defined integration measures on them still persists. Generalized integration theory [19] and *Radon-type of measures* on vector sheaves similar to the aforesaid 'cylindrical' ones employed by Ashtekar and Lewandowski (using Gel'fand's spectral theory) in the context of the holonomy C^*-algebraic approach to canonical quantum general relativity [6, 7, 8], are currently under intense development by ADG-theoretic means [71]. Such measures are expected to figure prominently in (and make mathematical sense of) heuristic (path) integrals like (136)–(139) and, in the finitary case, like (127) and (128).[261]

[258] It must be stressed that, according to the geometric (pre)quantization axiomatics [68, 69, 72, 73, 71] that we subjected our causon field $\vec{\mathcal{D}}_i$, or better, its associated fcqv-dynamo E-L field $(\vec{\mathcal{E}}_i^{\uparrow}, \vec{\mathcal{D}}_i)$ in [75], we can identify the latter with its quanta ('particles')—the causons (*eg*, states of 'bare' or free causons, when regarded as bosons—the 'carriers' of the dynamical field of quantum causality, are represented by sections of line bundles $\vec{\mathcal{L}}_i$ associated with the $\vec{\mathcal{P}}_i^{\uparrow}$s [75]). Thus, one can also think of $\overrightarrow{\mathcal{A}ut}_i$ as acting directly on the dynamical quanta of quantum causality—the causons. Shortly, we will revisit some basic geometric (pre)quantization arguments from [72] in order to further support these remarks.

[259] We are tempted to call our scheme, after Einstein, *'unitary' field theory*, since all that there is in it are the dynamical fields (plus their associated quanta and their automorphisms) and no ambient, external spacetime present. Because we have formulated gravity purely gauge-theoretically (*ie*, as the dynamics solely of the connection), we may alternatively coin our scheme *'pure gauge' field theory*.

[260] But note that in these works the incidence algebras are of a topological, not a directly causal, nature.

[261] Indeed, of special interest to ADG is to develop a general and mathematically sound integral calculus on the moduli spaces of gauge-equivalent connections on vector sheaves (those in particular that appear in the ADG-

2. *The problem of time*: again in the context of canonical quantum general relativity, this refers to the problem of requiring that the dynamics is encoded in the action of $\mathrm{Diff}(M)$ on the (Hilbert) space of physical states. Here too, our evasion of this problem is rather immediate:

$$\boxed{\text{No smooth space}\textit{time}\text{ manifold } M} \Rightarrow \boxed{\text{No Diff}(M)} \Rightarrow \boxed{\text{No problem of time}} \tag{142}$$

For, as we have repeatedly argued above, our theory deals directly with the dynamical physical objects $(\vec{\mathcal{D}}_i, \vec{\mathcal{E}}_i^{\uparrow})$ themselves and their (self-)transformations ('structure symmetries') $\overrightarrow{\mathcal{A}ut}_i$, and does not posit the existence of an external (background) spacetime continuum, let alone regard the latter as being physically significant in any way.[262] In our scheme, $\overrightarrow{\mathcal{A}ut}_i$ acts directly, via its representations alluded to in 1 above, on the associated (line) sheaves of bare causon states [93].[263]

3. *The problem of 'full covariance'*: as in 2 above, this problem essentially comes from assuming that the external, background spacetime manifold is a physical entity—and not paying attention just to the dynamical objects (fields and their particles) that live on that 'spacetime' which, anyway, are the only 'physically real' ('observable') entities. One is tempted to say here that the reason for this (problem) was in effect the lack of having thus far an appropriate framework to develop differential geometry—at least to the extent that ADG for instance has developed—different from that of the classical theory. In this respect, we may still recall here Einstein's 'confession' in [36]:

> *"...Adhering to the continuum originates with me not in a prejudice, but arises out of the fact that I have been unable to think up anything organic to take its place..."*

which we will mention again in 6.1 in connection with the singularities that assail the classical theory. In other words, the desirable scenario here is

> *the formulation of the (quantum) gravitational dynamics solely in terms of the connection* $\mathcal{D}$, or more completely, *in terms of the 'full', 'unitary' or 'pure' E-L field* $(\mathcal{E}^{\uparrow}, \mathcal{D})$*, and nothing else—in particular, without referring to an external (background) spacetime (whether the latter is assumed to be discrete or a continuum).*

theoretic treatment of Y-M theories and gravity [67, 68, 70]) again, *independently of the classical, differential manifold-based, theory* [71]. Such an abstract or generalized integration theory could be regarded as the ADG-theoretic analogue of the generalized integration and measure theory that has been developed (albeit, still in the $\mathcal{C}^\infty$-context!) in [11, 12].

[262] The reader should refer to the concluding section where further criticism is made of the base spacetime manifold M and its differentiable automorphisms $\mathrm{Diff}(M)$, as both are regarded as the last relics of an absolute, ambient, inert (non-dynamical), ether-like substance.

[263] See further remarks on geometric (pre)quantization that follow shortly.

As we saw earlier, in the canonical (Hamiltonian) approach to quantum general relativity there is a manifest breaking of covariance by the necessary $3+1$ dissection of the (external) spacetime continuum into space and time. Also, in a supposedly covariant path-integral-type of quantization scenario for Lorentzian gravity like (138) or (139), although there is no such an explicit external space-time split, there still persists however (built into the very CDG-formalism employed) the assumption of an external (background) geometrical M experiencing, for instance, problems like 1.[264]

5.4.2 A brief note on geometric (pre)quantization

Now that we have argued about how our theory can evade completely the inner product (Hilbert space) problem and the problem of time essentially by avoiding altogether the background M and its 'structure group' Diff(M), as well as how it may used to formulate a 'fully covariant' (quantum) dynamics for finitary and causal vacuum Einstein Lorentzian gravity, we would like to say a few words about another concrete application of ADG which further supports those arguments. This application concerns the subject of the so-called *Geometric (Pre)quantization* (GPQ) [68, 69, 72].[265]

We read from [72] that the main aim of GPQ is to arrive at a quantum model of a relativistic particle—which is assumed to be in the spectrum (*ie*, a so-called quantum particle excitation) of a corresponding quantum field—*without having to first quantize the corresponding classical mechanical system* [110]. In other words, GPQ aspires to a quantum description of elementary particles by referring directly to their ('second quantized') fields (*ie*, without the mediation of the procedure of first quantization of the classical mechanical or field theory and of the conventional Hilbert space formalism that accompanies it). On the other hand, it is well known that GPQ heavily rests on the usual differential calculus of $\mathcal{C}^{\infty}$-smooth (symplectic) manifolds[266]; hence, it is no surprise that ADG could be used to generalize the foundations of GPQ, thus gain more insight into the theory.

For instance, as we witnessed above, ADG completely circumvents the underlying $\mathcal{C}^{\infty}$-smooth spacetime manifold and deals directly with the (algebraic) objects that live on 'it'. These objects are the dynamical fields themselves (without recourse to an external base spacetime manifold) or equivalently, in a purely second quantized sense, the elementary particles (quanta) of these fields. In fact, the main objective of applying ADG-theoretic ideas to GPQ, basically motivated by certain fiber bundle axiomatics originally laid down by Selesnick in [102], is to show that

elementary particles—the quanta of the dynamical fields—can be classified according

[264] Let alone that in the actual implementation and interpretation of the path integral as a dynamical transition amplitude in the kinematical (moduli) space of gravitation 4-connections, 'boundary 3-geometries', which break full covariance, are implicitly fixed at the end-points of the otherwise indefinite integral (see footnote 242).

[265] In what follows, we do not intend to present any technical details from [68, 69, 72]; rather, we would like to give a brief outline of certain syllogisms and results of this application that further vindicate the aforesaid evasion by our ADG-based theory of the three problems of the background spacetime manifold based quantum general relativity theories whether they are Hamiltonian (canonical) or Lagrangian (path integral). As noted in footnote 258, we gather results mainly from [72].

[266] See remarks of Isham from [55] in the concluding paragraph of this section on GPQ.

to their spin in terms of appropriate vector sheaves $\mathcal{E}$.

In this respect, the main result of ADG applied to GPQ is that

> *states of bare (free) bosons can be identified with local sections of line sheaves $\mathcal{L}$,*[267] *while states of bare (free) fermions with local sections of vector sheaves $\mathcal{E}$ of rank greater than* 1 [68, 69, 72].

In order to arrive at that result, the first author had to posit the following identifications, or better, make the following bijective correspondences ('equivalences'), which we readily read from [72]:

1. States of elementary particles can be associated with (local) sections of appropriate vector sheaves $\mathcal{E}$, the latter being provided in the classical theory by the sheaves of sections of vector bundles over the spacetime manifold M *à la* Selesnick [102].[268]

2. An elementary particle—the irreducible constituent of matter—corresponds uniquely, in a second quantized sense, to the quantum of a particle field[269]; one writes:

$$\boxed{\text{physical particle}} \longleftrightarrow \boxed{\text{particle field}} \tag{143}$$

3. *A field, hence its quanta (elementary particles), is completely determined by its states.* The latter, within the axiomatic framework of ADG, correspond to local sections of suitably defined vector sheaves $\mathcal{E}$. All in all, one writes

$$\boxed{\text{particle}} \longleftrightarrow \boxed{\text{field}} \longleftrightarrow \boxed{\text{states}} \longleftrightarrow \boxed{\text{local sections}} \longleftrightarrow \boxed{\text{vector sheaf}} \tag{144}$$

with the latter identification (local sections $\longleftrightarrow$ vector sheaf) being, as a matter of fact, a well known theorem in sheaf theory.[270]

4. In fact, as we saw earlier, by 'field' ADG understands the pair $(\mathcal{E}, \mathcal{D})$.[271]

[267] That is to say, vector sheaves of rank 1.

[268] By Selesnick's work [102], these bundles correspond to finitely generated projective modules over the topological algebra $\mathcal{C}^\infty(M)$ of the smooth spacetime manifold M. ADG's primitive assumption of a general structure sheaf $\mathbf{A}$ other than $\mathcal{C}^\infty_M$ generalizes Selesnick's bundles to vector sheaves $\mathcal{E}$ that are locally free $\mathbf{A}$-modules of finite rank, as we saw before.

[269] The notion of 'field' being regarded here as an irreducible (*ur*) element of the theory, in the same way that Einstein thought of it as "*an independent, not further reducible, fundamental concept*" [38].

[270] That is to say, any (vector) sheaf is completely determined by its (local) sections [67, 68]. In fact, in [67] this has been promoted to the following important slogan: "*a sheaf is its sections*". So, there is a very close physico-mathematical analogy lurking in (144): *in the same way that a sheaf is completely determined by its sections, an elementary particle—ie*, the quantum of a field—*is completely determined by its states.*

[271] This vector sheaf-theoretic conception of a field by ADG comes as an abstraction and vector sheaf-theoretic generalization of Manin's fiber bundle-theoretic definition of the Maxwell's field of electrodynamics as the pair $(\mathcal{L}_{Max}, \mathcal{D}_{Max})$ consisting of a $(U(1))$ connection $\mathcal{D}_{Max}$ on a line bundle $\mathcal{L}_{Max}$ of 'photon states' [79]. It is also

5. Finally, and very briefly, starting from work by Selesnick in [102], the first author was led to realize that one can model the collection of quantum states of free elementary particles by *finitely generated projective A-modules*[272] and then, depending on their spin, classify them to *free bosons* whose states comprise projective A-modules of rank 1, and *free fermions* having for states elements of projective A-modules of rank greater than or equal to 2.[273] Then, the transition to locally finite $\mathbf{A}$-modules $\mathcal{E}$ of finite rank (*ie*, the vector sheaves of ADG) was accomplished by using the Serre-Swan theorem (suitably extended from the Banach algebra $A = \mathcal{C}^0(M)$ on a compact Hausdorff manifold M to general topological non-normed (non-Banachable) algebras such as $\mathcal{C}^\infty(M)$) in order to go from the aforesaid finitely generated $\mathcal{C}^\infty(M)$-modules to smooth vector bundles on M. Then the latter can provide us with the (local) sections we need to build our $\mathcal{E}$s.

6. All in all, the general result of applying ADG to GPQ is the following 'categorical' statement [68, 69, 75]

> *every (free) elementary particle is (pre)quantizable (ie, it admits a (pre)quantizing line sheaf).*

It must be noted here that the sheaf-cohomological classification of our fcqv-E-L fields $(\vec{\mathfrak{D}}_i, \vec{\mathcal{E}}_i^{\uparrow})$ and their quanta (causons) in [75] is essentially an application of the results of the ADG-theoretic perspective on GPQ above to the finitary, causal and quantal regime. *In toto*, and this is the main reason we briefly alluded to ADG *vis-à-vis* GPQ here,

> *being able, by circumventing ADG-theoretically the classical external $\mathcal{C}^\infty$-spacetime manifold M, to refer directly to the dynamical objects (fields), we can show not only that (the dynamics of) these objects are 'fully covariant', but also that they are 'intrinsically' of a quantum nature,*[274] *so that the quest for a 'blindfolded', head-on quantization of spacetime and general relativity*[275] *appears to be begging the question. Indeed, since our scheme is 'fully covariant', 'inherently quantum'*[276] *and it certainly does*

important to remark here that, semantically, ADG regards the connection $\mathcal{D}$ as '*the dynamical field proper*', while $\mathcal{E}$ as '*the carrier (state) space of (the particles or quanta of) the field*'. In fact, both $\mathcal{D}$ and $\mathcal{E}$ are needed for formulating the laws of nature ('differential equations') as $\mathcal{E}$ provides us with the sections (states of the particle—the 'Being' of the particle so to speak) on which $\mathcal{D}$ acts (*ie*, dynamically transforms the particle—the 'Becoming' of the particle so to speak). It is conceptually lame, perhaps even 'wrong', from the ADG-theoretic perspective to think of $\mathcal{E}$ ('state') apart from $\mathcal{D}$ ('transformation of state') and vice versa. The concept of field in ADG, as the pair $(\mathcal{E}, \mathcal{D})$, is a 'holistic', 'unitary' or 'coherent' one, not separable or 'dissectible' into its two constituents.

[272] Finiteness pertaining to the finite dimensionality of the representations of the particles' compact structure (symmetry) gauge group.

[273] In particular, by taking A to be $\mathcal{C}^\infty(M)$ [72].

[274] That is, dealing directly and exclusively with the propagating field is equivalent to dealing directly and solely with its dynamical quantum (particle).

[275] That is, of the dynamics of the smooth gravitational field (whether this is represented by the metric or the connection-*cum*-frame field) propagating on a $\mathcal{C}^\infty$-spacetime manifold.

[276] In fact, we are tempted to regard these two characterizations of our theory (*ie*, 'fully covariant' and 'intrinsically quantum') as being equivalent, for ADG refers directly to the dynamical fields and their quanta. Some strong

> *not arise from 'quantizing somehow the classical theory', we strongly doubt whether actually quantizing a classical theory is physically meaningful at all.*[277] Thus, with respect to the ADG-based theory for fcqv-E-L gravity propounded here, to this last question whether quantizing a classical theory (in our case, general relativity) is physically meaningful at all, one might respond by remarking that *this always depends on the type of theory that one employs in order to describe the physical laws through the corresponding (differential) equations.*

The last remarks would strike one, who is used to the idea that one should be able to arrive at a quantum theory of gravity by quantizing somehow general relativity (*ie*, by employing a formal quantization procedure involving the usual quantum mechanical concepts and mathematical structures such as 'observables', Hilbert spaces *etc* while still retaining the classical calculus-based framework for both an external spacetime and the dynamical laws for the now quantized fields on it), as being at best odd, if we also quote the following passage from a celebrated textbook that has nurtured generations and generations of theoretical physicists [61]:

> "Quantum mechanics occupies a very unusual place among physical theories: it contains classical mechanics as a limiting case, yet at the same time *requires*[278] this limiting case for its own formulation."

the emphasized'*requires*' being here the 'operative word'—precisely the one we have challenged and doubted in the present paper.[279] For, as it was noted at the end of 5.3.2, we already have strong indications that trying to quantize head-on general relativity is perhaps not the right way to a quantum theory of gravity [43, 44, 56]. In a nutshell then, *we doubt that quantum gravity is,* or better, *will prove to be quantized gravity.*

We would like to close this discussion of the ADG-theoretic perspective on GPQ with some very pertinent remarks of Isham in his latest paper [55][280] which emphasize precisely how the (geometric) quantization of a classical theory is fundamentally (and quite *a priori*ly, *ad hoc*, thus inappropriately—especially for quantum gravity research) based on the classical differential geometry of smooth manifolds (essentially because the conventional quantum theory itself,

conceptual resonances with Einstein's vision of a unitary field theory (which can 'explain' quantum phenomena) are pretty obvious here.

277 For instance, since first quantization is totally bypassed by GPQ, there is *prima facie* no need for reasoning 'conventionally' (*ie*, by using Hilbert spaces, 'observables' and the rest of the conventional jargon, methods and technical baggage of quantum mechanics) about causons and their dynamics. In fact, the correspondence principle advocated initially in [92, 93] about the incidence algebras modelling discrete and quantum topological spaces should by no means be regarded as a 'consistency' or 'physicality check' of our theory (*ie*, as if our theory *should* yield classical gravity as a 'low energy or weak gravitational field limit' in the same way that the other discrete spacetime or continuum-based approaches to quantum gravity are expected to). From the purely ADG-theoretic point of view, immediate contact with the classical theory is established simply by setting $\mathbf{A} \equiv \mathcal{C}^\infty_M$.

278 Our emphasis.

279 In our case, one should substitute the word 'mechanics' by 'gravity' or even by 'general relativity' in the quotation above in order to get a better feeling of the point we wish to make. (Of course, this is an imaginary, 'wishful thinking' situation in which we are talking about quantum gravity as if it has already been formulated!)

280 The excerpts below are taken from 2.1.1 in [55].

which we apply when we wish to quantize a classical theory, is based on the manifold model for spacetime[281]).

> "...*In general,* [when we start from a classical theory and then 'quantise' it]*, the configuration space (if there is one)* Q *for a classical system is modelled mathematically by a differentiable manifold and the classical state space is the co-tangent bundle* T^*Q*. The physical motivation for using a manifold to represent* Q *again reduces to the fact that we represent physical space with a manifold...*
>
> *Thus, in assuming that the state space of a classical system of the form* T^*Q *we are importing into the classical theory a powerful a priori picture of physical space: namely, that it is a differentiable manifold.*[282] *This then carries across to the corresponding quantum theory. For example, if 'quantization' is construed to mean defining the quantum states to be cross-sections of some flat vector bundle over* Q*, then the domain of these state functions is the continuum space* Q*...*"

This is more or less how (second) 'quantization' was originally construed fiber bundle-theoretically in [102] and then was treated ADG-theoretically to suit GPQ ideas—albeit, in the characteristic absence of a $\mathcal{C}^\infty$-smooth base spacetime continuum (domain)—in [68, 69, 72] and, in the finitary spacetime and gravity case, in [75]. From this point of view, this is another indication that our finitistic theory for vacuum Einstein-Lorentzian gravity here may be regarded as being 'already quantized' (better, 'inherently quantum')—albeit, not at all 'conventionally' in Isham's sense of the word (which means that one applies the usual quantum theory, with its classical manifold conception of space and time, to an already existing classical theory).

5.4.3 Remarks on Einstein's 'new ether' and unitary field theory vis-à-vis 'full covariance'

Here we would like to bring together certain ideas that were expressed above—in particular, in connection with the full covariance of our theory, the identifications (143) and 144) in the context of geometric (pre)quantization, as well as with some allusions made earlier to our hunch that our scheme is 'already quantum', as it were, not in need of quantizing (*ie*, applying quantum theory to) the classical theory of gravity (general relativity)—and some of Einstein's searching thoughts about a new conception of 'ether' in the light of his continuous unitary field theory, singularities and the quantum paradigm.[283] We will see how Einstein (i) tried to respect as much as he could

[281] See again related comments in our discussion of the use of $\mathbb{R}$ and $\mathbb{C}$ in our theory in 5.1.

[282] "There may be cases [like those arising in the context of geometric quantization theory] where $\mathcal{S}$ is a symplectic manifold that is not a cotangent bundle; for example, $\mathcal{S} = S^2$. However, I would argue that the reason $\mathcal{S}$ is assumed to be a *manifold* is still ultimately grounded in an *a priori* assumption about the nature of physical space (and time)." (Our addition in square brackets.)

[283] By unitary field theory we do not refer so much to the more well known, life-long endeavor of Einstein to unify gravity with electromagnetism and regard material particles as being special states of condensed energy of (*ie*, 'singularities' or 'discontinuities' in) the (continuous) unified field [17], as to his general intuition—which is of course closely related to his well known unitary field theory project—that all physical actions (including quantum matter) must be described in terms of (continuous) fields. However, below we are also going to comment on unified field theory in the more popular sense of the term.

general relativity which posits an ether-like spacetime background in the form of the differential manifold and the smooth metric field imposed on this spacetime continuum, (ii) always kept in mind the earlier abolition of the material ether by special relativity so that he was careful not to attribute mechanical properties to the ambient geometrical spacetime continuum,[284] and (iii) was deeply impressed by the discontinuous actions of (matter) quanta, and he intuited—at times in an 'oxymoronic' way which reflects precisely the opposite tension in his mind between the continuous/geometrical actions of (special and) general relativity and the discrete/algebraic ones of quantum theory—a new kind of 'ether' intimately related to the spacetime continuum which may be cumulatively referred to as '*the continuous unitary field*'. Then, we will discuss the affinities and the fundamental differences between the latter, continuum spacetime metric field-based (geometrical) and our ADG-theoretic, connection-based 'fully covariant' and 'inherently quantum' (reticular-algebraic) vacuum Einstein-Lorentz gravity. Along with the Einstein references at the back, in the sequel we borrow some of Einstein's quotations and various ideas about this rebirth of the notion of ether from [60].

We commence with a quotation of Einstein, as early as 1924, in which, in spite of the abolition of the 'material' and 'mechanical' luminipherous ether by the special theory of relativity already almost two decades earlier, he insists that in the context of a continuous field theory on a spacetime continuum the notion of ether (even if a generalized, non-mechanistic or non-material one) is physically quite indespensible. For example, he concludes the article '*Über den Äther*' [34] as follows:

> "*...But even if these possibilities should mature into genuine theories, we will not be able to do without the ether in theoretical physics, ie, a continuum which is equipped with physical properties; for the general theory of relativity, whose basic points of view surely will always maintain, excludes direct distant action. But every contiguous action theory presumes continuous fields, and therefore also the existence of an 'ether'.*"[285]

Therefore, for Einstein, the spacetime continuum, supporting continuous fields, provides a *new ether* paradigm. At the same time, he readily and repeatedly denied the independent physical existence of space(time) apart from the continuous field and the (in his own words) 'physical continuum' (*ie*, the ether) that supports or 'carries' it, much as follows:

> "*...According to general relativity, the concept of space detached from any physical content does not exist. The physical reality of space is represented by a field whose components are continuous functions of four independent variables—the coordinates of space and time. It is just this particular kind of dependence that expresses the spatial character of physical reality.*" [60][286]

and

[284] In a sense, field theory is not mechanistic.

[285] While, already four years earlier [32], he had stressed the 'ether imperative' in physics as follows: "*...The ether hypothesis must always play a part in the thinking of physicists, even if only a latent part.*".

[286] Page 175 and reference therein.

"*...If the laws of this field are in general covariant, that is, are not dependent on a particular choice of coordinate system, then the introduction of an independent (absolute) space is no longer necessary. That which constitutes the spatial character of reality is simply the four-dimensionality of the field. There is no 'empty' space, that is, there is no space without a field.*" [60][287]

and, in a sense that was emphasized throughout the present paper, he essentially maintained that *(the) space(time) continuum and, concomitantly, the (new) ether is inherent in the (gravitational) field*[288]:

"*...No space and no portion of space can be conceived of without gravitational potentials; for these give it its metrical properties without which it is not thinkable at all. The existence of the gravitational field is directly bound up with the existence of space...*" [32]

also

"*...according to the general theory of relativity even empty space has physical qualities, which are characterized mathematically by the components of the gravitational potential.*" [60][289]

and

"*Thus, once again 'empty' space appears as endowed with physical properties, i.e., no longer as physically empty, as seemed to be the case according to special relativity. One can thus say that the ether is resurrected in the general theory of relativity, though in a more sublimated form.*" [60][290]

furthermore

"*...There is no such thing as empty space, i.e., a space without field. Space-time does not claim existence on its own, but only as a structural quality of the field...*" [37]

and

"*...space has lost its independent physical existence, becoming only a property of the field...*" [38][291]

[287] Again, page 175 and reference therein.

[288] Which, unlike in our algebraic, connection-based theory however, he identified with (the components of) the metric tensor $g_{\mu\nu}$.

[289] Again, page 111 and reference therein.

[290] Page 111 and reference therein.

[291] This brings to mind the remarks, albeit in the context of the flat spacetime (quantum) field theory of matter, of Denisov and Logunov: "*...Minkowski was the first to discover that the space-time, in which all physical processes occur, is unified and has a pseudo-Euclidean geometry. Subsequent study of strong, electromagnetic, and weak interactions has demonstrated that the pseudo-Euclidean geometry is inherent in the fields associated with these interactions...Pseudo-Euclidean space-time is not a priori, i.e., given from the start, or having an independent existence. It is an integral part of the existence of matter,...it is [always] the geometry by which matter is transformed...*" [25]. Indeed, back in 5.1.1, and shortly in our comments on Gel'fand duality (5.5.1), we argue how the geometrical structure of what one might call 'spacetime' (including its topology and differential structure) is inherent in the algebraic-dynamical field of quantum causality in the same way that the geometrical notion of curvature is already inherent (ultimately, derives from) the dynamical connection field, which is the sole physically meaningful entity in our theory.

while, for the sake of operationality or instrumentality (*ie*, for the existence of measuring rods and clocks)[292]

> "*...According to the general theory of relativity, space without ether is unthinkable; for in such space, not only would there be no propagation of light, but also no possibility of existence for standards of space and time (measuring rods and clocks), nor therefore any space-time intervals in the physical sense...*" [32]

Thus, eventually, he was led to make the following (telling for us) conceptual identification:

> "*...Physical space and the ether are only different expressions for one and the same thing...*" [60][293]

Moreover, keeping the identification above in mind, we note that Kostro, in [60][294], expresses concisely how this new ether may culminate in the formulation and serve as the basic underlying concept of a unified field theory (in the more popular sense), as follows:

> "*The last step in the development of the relativistic concept of the ether would be the creation of a unified field theory in which a unification of gravitational and electromagnetic interactions is achieved and in which matter consisting of particles would constitute special states of physical space. Thus far, the attempts to develop such a theory have been unsuccessful, the reason lying not in physical reality, but in the deficiencies of our theories. It would be ideal to develop such a unified field theory in which all the objects of physics would come under the concept of the ether. Einstein pointed out this problem at the very beginning of his article:*[295]
>
> *...one can defend the view that this notion* [*ie*, the ether] *includes all objects of physics, since according to a consistent field theory, ponderable matter and the elementary particles from which it is built also have to be regarded as 'fields' of a particular kind or as particular 'states' of space.*"

This prompts us to cast, in complete analogy to the ADG-theoretic identifications in the context of geometric (pre)quantization in (143) and (144), Einstein's conceptual identifications above as a *résumé* of his unitary field theory program, as follows:

$$\begin{array}{ccccc} \boxed{\text{elementary particles/matter quanta}} & \longleftrightarrow & \boxed{\text{states of the continuous unitary field}} & \longleftrightarrow & \boxed{\text{states of the spacetime continuum}} \\ & & \Big\| & & \\ & & \boxed{\text{states of the new ether}} & & \end{array} \tag{145}$$

[292]And this shows just how important for the physical interpretation of the theory Einstein thought the operational foundations of general relativity are.

[293]Page 174 and reference therein.

[294]Bottom of page 105 and top of page 106.

[295]Einstein's article Kostro is referring to is '*Über den Äther*' [34].

In comparison with our identifications in (144), we note that since our ADG-theoretic perspective on finitary, causal and quantal vacuum Einstein-Lorentzian gravity completely evades the smooth background spacetime continuum and is based solely on the fcqv-E-L field $\vec{\mathcal{D}}_i$, *our (arguably more quantal, because reticular-algebraic) version of Einstein's new ether above could be taken to be the 'carrier' of this causon field, namely, the vector sheaf* $\vec{\mathcal{E}}_i^{\uparrow}$ *itself. The latter*, in close analogy to the inextricable relationship between the ether, the (continuous) space(time) and the (gravitational) field that Einstein intuited, but with the prominent absence of an external, background $\mathcal{C}^\infty$-spacetime and our undermining of the physical role played by the smooth gravitational metric field $g_{\mu\nu}$ supported by it, *cannot be thought of independently of the fcqv-gravitational connection that it carries and vice versa*.[296]

Now, since Einstein was well aware of the problem of singularities that plague his geometric spacetime continuum based theory of gravity[297], and at the same time he was 'in awe' of the (successes of the) quantum revolution, he on the one hand asked:

> "*Is it conceivable that a field theory permits one to understand the atomistic and quantum structure of reality?*" [38]

and on the other, quite paradoxically if we consider the conceptual importance that he placed on the continuous field and the spacetime continuum (*ie*, the new ether) supporting it, he repeatedly doubted in the algebraic light of the quantum the very geometrical ether (*ie*, the $\mathcal{C}^\infty$-smooth spacetime continuum and the smooth metric field $g_{\mu\nu}$ that it supports) that he so feverously propounded in the quotes above.[298] For instance, until the very end of his life he doubted the harmonious coexistence of the (continuous) field together with its particles (quanta) in the light of the singularities that assail the spacetime continuum, much as follows:

> "*...Your objections regarding the existence of singularity-free solutions which could represent the field together with the particles I find most justified. I also share this doubt. If it should finally turn out to be the case, then I doubt in general the existence of a rational and physically useful continuous field theory. But what then? Heine's classical line comes to mind: 'And a fool waits for the answer'...*" (1954) [119]

How can we explain and understand this apparently 'paradoxical' and 'self-contradictory' stance of his against the spacetime continuum *vis-à-vis* singularities and the quantum?[299] Perhaps

[296] See again footnote 271 about this 'holistic' or, quite fittingly, 'unitary' ADG-theoretic conception of the gravitational connection and the vector sheaf (of states of causons in our finitary theory) that carries it—our version of Einstein's 'new ether'.

[297] See quotations of Einstein subsequently and our discussion in the epilogue.

[298] See quotations in 5.1 and more extended ones in [74, 75].

[299] That is, on the one hand to argue for the geometrical spacetime continuum, in the guise of the new ether, which is inherent in the continuous unitary field representing the field together with its quanta—the particles that may in turn be conceived as 'singularities in the field', and at the same time on the other, exactly due to those singularities (*eg*, the infinities of fields right at their point-particle 'sources') of the manifold and the discontinuous, algebraically represented actions of quanta, to urge us to abandon the geometrical continuous field theory and look for "*a purely algebraic theory for the description of reality*" [38, 74]—one whose "*statements are about a discontinuum without calling upon a continuum space-time as an aid*" and according to which "*the continuum space-time construction corresponds to nothing real*" (1916) [119, 75].

we can understand his apparently 'circular' and 'ambiguous' attitude if we expressed the whole 'oxymoron' in a positive way, as follows: *we believe that Einstein would have readily abandoned the continuous field theory and the geometric spacetime continuum of general relativity in view of the 'granular' actions of quantum theory if he had an 'organic'*[300] *finitistic-algebraic theory to take its place*. Alas, again in his own words just a year after he concluded the general theory of relativity and at the very end of his life:

> "*...But we still lack the mathematical structure unfortunately.*" (1916)[301]

and

> "*...But nobody knows how to obtain the basis of such a* [finitistic-algebraic] *theory.*" (1955)[302]

We would like to think that ADG, especially in its particular finitistic-algebraic application here to Lorentzian vacuum Einstein gravity, goes some way towards qualifying as a candidate for the (mathematical) theory that Einstein was searching for. Since we are talking about Einstein's unitary field theory and the mathematics that he was searching for in order to implement it, we give below a very fitting passage which concludes Ernst Straus' reminiscences of Einstein in [121]:

> "*...Einstein's quest for the central problem for the ultimate correct field theory is generally considered to have failed. I think that this did not really surprise Einstein, because he often entertained the idea that vastly new mathematical models would be needed, that possibly the field-theoretical approach through the kind of mathematics that he knew and in which he could do research would not, could not, lead to the ultimate answer,*[303] *that the ultimate answer would require a kind of mathematics that probably does not yet exist and may not exist for a long time. However, he did not have the slightest doubt that an ultimate theory does exist and can be discovered.*"

We sum-up this discussion of Einstein's new ether by commenting on and counterpointing some remarks of Peter Bergmann and Ludwik Kostro in [60][304] which apparently maintain that what Einstein had in mind when he talked about this new ether in the context of unitary field theory was not the $\mathcal{C}^\infty$-smooth spacetime manifold *per se*, but the extra structures (such as the metric, for example) that are imposed on it.

First, Kostro asked:[305]

---

[300] See quotation from [36] and in 6.1.
[301] For the whole quotation, see [75].
[302] This is the last sentence, in the last section of the last appendix of '*The Meaning of Relativity*' [38] appended in 1954. The whole quotation can be found directly at the end of [74].
[303] See remarks by Bergmann and Kostro that follow shortly; especially Kostro's words in footnote 311 about the mathematics that Einstein knew and used in order to model his unitary field theory.
[304] Pages 164 and 165.
[305] In a talk titled '*Outline of the history of Einstein's relativistic ether conception*' delivered at the International Conference on the History of General Relativity, Luminy, France (1988) [60].

> "*...Which mathematical structure of contemporary theoretical physics represents the entity Einstein called 'the new ether'?*"

to which Bergmann replied:

> "*...In the last decades of his life Einstein was concerned with unitary field theories of which he created a large number of models. So I think he was very conscious of the distinction between the differential manifold (though he did not use that term)*[306] *and the structure you have to impose on the differential manifold (metric, affine or otherwise) and that he conceived of this structure, or set of structures, as potential carriers of physical distinctiveness and including the dynamics of physics.*
>
> *Now, whether it is fortunate or unfortunate to use for the latter the term like ether? I think simply from the point of view of Einstein and his ideas that in the distinction between the differential manifold as such and the geometrical structures imposed on it we could, if we want, use the term ether for the latter.*"

and to which, in turn, Kostro further added:

> "*I am certain that Bergmann was right when he claimed that the differential manifold as such, which is used to model space-time without imposing upon it such structures as metrics, etc. cannot be treated as a mathematical structure representing Einstein's relativistic ether.*
>
> *Bergmann was right, because the four-dimensional differential manifold as such is a mathematical structure of too general a nature, and it cannot physically define distinctive features of the space-time continuum without imposing metrics and other structures upon it. It is too general, because it can serve as an arena or background for any macroscopic physical theory (and even perhaps a microscopic one, because the debate over the status of the differential manifold in microphysics is ongoing). By the act of imposing metrics (i.e., the recipe for measuring space and time intervals) and other structures upon it, the structure enriched in such a way turns into something that represents distinctive physical features of the real space-time continuum...*"

We partially agree with Bergmann and Kostro insofar as their comments above entail that the background differential spacetime manifold itself is devoid of physical significance and that what is of physical importance is the 'geometrical' objects that live on this base arena which, in Bergmann's words, "*include the dynamics of physics*". On the other hand, from the novel perspective of ADG, and we would like to believe that both Bergmann and Kostro would agree with us had they been familiar with the basic tenets of ADG, we maintain that:

1. In general relativity, the smooth spacetime manifold serves as the *carrier* of the structures imposed on it—after all, this is how the structures like metric, affine (Levi-Civita) connection *etc* acquire the epithet '*smooth*' in front and become *smooth metric*, *smooth connection*

[306] Einstein most of the time used the term '*(space-time) continuum*' (our footnote).

etc.[307] As such, *it can still be perceived as a passive, a priori fixed by the theorist, absolute, ether-like substance which sets the classically unequivocal 'condition or criterion of differentiability' for the dynamical variations of these 'physical' structures imposed on it.*[308] For, surely, if Einstein did not have the background $\mathcal{C}^\infty$-spacetime at his disposal, the (classical) differentials that the latter provides one with and the rules of the mathematical theory known as (classical) differential geometry (calculus) of manifolds that these differentials obey, how could he write the dynamical laws for the aforesaid extra physical structures? And, arguably, in a Wheelerian sense, *no theory is a physical theory unless it is a dynamical theory*. Thus, the usual differential calculus provided Einstein with the basic mathematical tools which enabled him to write the dynamical equations for his continuous, 'ethereal' fields.

2. As noted above, one should not forget that Einstein's dissatisfaction with the geometrical spacetime continuum (manifold) came basically from two sources: the singularities that assail general relativity and, of course, the discontinuous and algebraic character of quantum mechanical actions. In fact, at the very end of his life, and in the context of his unitary field theory, he came to intuit that these two 'problematic', when viewed from the spacetime continuum perspective, sources may be in fact intimately related:[309]

   > "*...Is it conceivable that a field theory permits one to understand the atomistic and quantum structure of reality? Almost everybody will answer this question with 'no'. But I believe that at the present time nobody knows anything reliable about it. This is so because we cannot judge in what manner and how strongly the exclusion of singularities reduces the manifold of solutions. We do not possess any method at all to derive systematically solutions that are free of singularities...*"

   ADG, as applied here (and in [74, 75]) to a locally finite, causal and quantal vacuum Einstein gravity, 'kills both birds above with one stone': on the one hand, it evades the $\mathcal{C}^\infty$-manifold and 'engulfs' or 'absorbs' singularities into whichever structure sheaf of generalized arithmetics (or coordinates) one chooses to employ in order to tackle the physical problem one wishes to study [78, 73, 76],[310] and on the other, it offers us an entirely algebraic and finitistic way of doing (the entire spectrum of the usual) differential geometry [67, 68, 77, 74, 75, 71]. All in all, it is our contention that Einstein (implicitly) questioned the very (pseudo-)Riemannian differential geometry, which, in turn, fundamentally relies on the differential spacetime manifold.

[307] With the important clarification, however, that it is a rather common mistake (made nowadays especially by theoretical physicists) to think that the metric was assigned (originally by Gauss and Riemann) on the manifold itself. Rather, it was imposed on (what we now call) the (fibers of the) tangent bundle (tangent to whatever 'space' they used as base space)! (revisit footnote 17). Thus, the commonly used term '*spacetime metric*' can be quite misleading.

[308] See our comments on the relativity of differentiability in the epilogue.

[309] The following quotation can be found again in the last appendix of [38]. It is the extended version of the one given a few paragraphs above.

[310] Again, for more comments on singularities, the reader should go to the epilogue of the present paper.

3. From the ADG-based perspective of the present paper, *there is nothing physical about either an external background spacetime (be it discrete or continuous) or about the metric structure that we impose on it*. On these grounds alone, Bergmann and Kostro's contention above that these concepts may be regarded as representing Einstein's new ether, appears to be unacceptable. On the other hand, we believe that our entirely algebraic conception of the (gravitational) connection can be seen as the sole dynamical variable in a quantal theory of Lorentzian gravity. Fittingly then, the (associated) vector sheaf (of states of causons), which are not soldered (*ie*, localized) on any $\mathcal{C}^\infty$-smooth spacetime manifold whatsoever, may be taken to be as the ADG-theoretic analogue of Einstein's 'new ether': it is the carrier of the fcqv-E-L field.

4. Finally, in view of the words of Feynman and Isham in the beginning of the present work, as well as what has been shown, partially motivated by these (or rather, 'post-anticipatorily'), in the present paper, *we simply have to disagree with Kostro's contention that there is still a possibility that the smooth manifold can serve as a (spacetime) background for a micro-physical theory—in particular, in the (feverously sought after) quantum theory of gravity.* Although, admittedly, Einstein did not know and use the differential geometry of smooth manifolds the way we do today (*eg*, fiber bundle theory),[311] he still had the tremendous physical insight to anticipate and foreshadow subsequent thinkers and workers in quantum gravity, like Feynman and Isham for example, who have been led by their own quests to conclude that *the $\mathcal{C}^\infty$-smooth model of spacetime fares poorly*, to put it mildly,[312] *in the quantum (gravity) regime.*

5.4.4 Brief remarks on 'the matter of the fact'

Since we have just commented on Einstein's unitary field theory, since in causet theory there has been a strong indication lately that one can derive matter fields directly from causets [95], and also since our scheme so far has focused solely on pure vacuum gravity (*ie*, without the inclusion of matter actions and other gauge force fields), we would like to conclude this subsection by making a very short comment on the possibility of including matter and other gauge field actions in our locally finite, causal and quantal theory. Our brief addendum is simply that, *prima facie*, the inclusion of fermionic matter fields (*eg*, electrons), their connections (*eg*, Dirac-like operators), as well as their relevant gauge potentials (*eg*, electromagnetic field), can be straightforwardly implemented ADG-theoretically as follows:

1. In line with our comments earlier on GPQ, the (states of) other gauge (boson) and matter (fermion) fields can be modelled by (local) sections of the relevant line (rank $=1$) and vector (rank >1) (fin)sheaves (here, over a causet), respectively.

[311] And at this point we agree with Kostro when he says that "*...In the physical space-time continuum model in his Special Theory of Relativity and General Theory of Relativity, and in his attempts to formulate a unitary relativistic field theory, Einstein could not apply the tools and methods of the contemporary theory of differential manifolds and the structures we use with them, because he simply did not know them in the form in which they are taught and applied today...*" ([60], p. 164).

[312] Not to say 'fails miserably'.

2. Their corresponding (gauge) connections will be modelled by their relevant finsheaf morphisms, and their (free) dynamics by equations involving (the field strengths of) these morphisms (which dynamics, in turn, by the very categorical definition of those finsheaf morphisms and the covariance of their corresponding field strengths, will be manifestly gauge $\mathcal{U}_i$-invariant).

3. Interactions between the matter and their gauge fields will be algebraic expressions involving the relevant finsheaf morphisms coupled to (*ie*, acting on) the aforesaid (local) sections.

4. *In toto*, in the finitary case of interest here, 1–3 will be finitistic, causal, explicitly independent of an external, underlying (*ie*, background) $\mathcal{C}^\infty$-smooth spacetime continuum (*ie*, 'fully covariant'), 'purely gauge-theoretic' and 'inherently quantum', as it was the case for the vacuum gravitational field elaborated in the present paper.

However, for more information about the general ADG-theoretic treatment of (non-gravitational) gauge (*ie*, electromagnetic and non-abelian Yang-Mills) theories and their quantum matter sources, the reader should refer to [71].

5.5 Comments on Gel'fand Duality and the Power of Differential Triads

We close the present section by commenting briefly on the notion of Gel'fand duality—an idea that we repeatedly alluded to and found of great conceptual import in the foregoing. We also illustrate how powerful the basic ADG-theoretic notion of differential triads is for establishing continuum ('classical') limits for a host of (physically) important mathematical structures that we encountered earlier during the *aufbau* of our locally finite, causal and quantal vacuum Einstein gravity.

5.5.1 Gel'fand duality: from algebras to geometric spaces and back

By Gel'fand duality we understand the general 'functional philosophy' according to which, informally speaking,

the variable (argument) becomes function and the function variable (argument).

One could symbolically represent this as follows

$$f(x) \longrightarrow \hat{x}(f) \tag{146}$$

For example, in the previous section we noted that our work with (finsheaves of) incidence algebras associated with (over) the finitary topological posets of Sorkin is essentially based on Gel'fand duality so that, in discussing inverse and inductive limits of those posets and (the finsheaves of) their incidence algebras respectively, we ended up concluding that *'space(time)' is categorically or Gel'fand dual to the physical fields that are defined on 'it'*. This is precisely the

semantic content of (146), but let us explicate further this by starting from the classical and well understood theory.

From the classical manifold perspective, Gel'fand duality has an immediate and widely known application: the (topological) reconstruction of a $\mathcal{C}^\infty$-smooth manifold M as the spectrum $\mathfrak{M}$ of its algebra $\mathcal{C}^\infty(M)$ of smooth functions [65]. To describe briefly this, let M be a differential manifold and x one of its points. Consider then the following collection of smooth $\mathbb{R}$-valued functions on M

$$I_x = \{\phi : \ M \longrightarrow \mathbb{R} |\ \phi(x) = 0\} \subset {}^{\mathbb{R}}\mathcal{C}^\infty(M) \tag{147}$$

It is straightforward to verify that I_x is a maximal ideal of ${}^{\mathbb{R}}\mathcal{C}^\infty(M)$ and that the quotient of the latter by the former yields the reals: ${}^{\mathbb{R}}\mathcal{C}^\infty(M)/I_x = \mathbb{R}$. In fact, in complete analogy to the space $\mathrm{Max}(\mathfrak{C})$ that we encountered earlier in connection with Ashtekar and Isham's commutative C^*-algebraic approach to the loop formulation of canonical quantum gravity which employs the Gel'fand-Naimark representation theorem[313]—it too a straightforward application of Gel'fand duality,[314] the set $Spec[{}^{\mathbb{R}}\mathcal{C}^\infty(M)] \equiv \mathfrak{M}[{}^{\mathbb{R}}\mathcal{C}^\infty(M)]$ of all maximal ideals I_x $(x \in M)$ of ${}^{\mathbb{R}}\mathcal{C}^\infty(M)$ such that

$$\mathbb{R} \hookrightarrow {}^{\mathbb{R}}\mathcal{C}^\infty(M) \longrightarrow {}^{\mathbb{R}}\mathcal{C}^\infty(M)/I_x \tag{148}$$

(within an isomorphism of the first term), is called *the real (Gel'fand) spectrum of* ${}^{\mathbb{R}}\mathcal{C}^\infty(M)$. Furthermore, if ${}^{\mathbb{R}}\mathcal{C}^\infty(M)$—regarded algebraic geometrically as a commutative ring—is endowed with the so-called Zariski topology [49], or equivalently, with the usual Gel'fand topology,[315] then the 'point-wise' map

$$M \ni x \mapsto I_x \in \mathfrak{M}[{}^{\mathbb{R}}\mathcal{C}^\infty(M)] \tag{149}$$

can be shown to be a homeomorphism between the $\mathcal{C}^0$-topology of M (*ie*, M being regarded simply as a topological manifold and the Gel'fand (Zariski) topology of $\mathfrak{M}[{}^{\mathbb{R}}\mathcal{C}^\infty(M)]$. *In toto*, the essential idea of Gel'fand duality here is to substitute the (topology of the) underlying space(time) continuum by the (algebras of) objects (functions/fields) that live on it, and then recover it by a suitable technique, which we may coin *Gel'fand spatialization*.

[313]It must be noted however that ${}^{\mathbb{R}}\mathcal{C}^\infty(M)$ is an abelian *topological algebra*, not a Banach, let alone a C^*-, algebra. In point of fact, it is well known that ${}^{\mathbb{R}}\mathcal{C}^\infty(M)$ is not 'normable' or 'Banachable' (Šilov) [65]. On the other hand, ${}^{\mathbb{C}}\mathcal{C}^0(M)$, for a compact manifold M, is the 'archetypal' commutative C^*-algebra—the very one Ashtekar and Isham used in [6] to represent $\mathfrak{C}$.

[314]For example, the Gel'fand transform in (139) is a precise mathematical expression of a Gel'fand duality between the space of connections and the space of loops involved in that theory [6, 8]. Furthermore, to 'justify' the notation in (146), we note how in [68] the Gel'fand transform is defined (in the case of a topological algebra A): let A be a (unital, commutative, locally m-convex) topological algebra, whose spectrum (*ie*, the set of non-zero, continuous, multiplicative linear functionals on A) is $\mathfrak{M}(A)$. The latter is equipped with the usual Gel'fand topology relative to which the maps $\hat{x} : \ \mathfrak{M}(A) \longrightarrow A$, with $\hat{x}(f) := f(x)$, are continuous. Then, the Gel'fand transform algebra of A, is defined as: $A\hat{} := \{\hat{x} : \ x \in A\}$.

[315]The coincidence between the Gel'fand and the Zariski topology on $\mathfrak{M}[{}^{\mathbb{R}}\mathcal{C}^\infty(M)]$ is due to the fact that ${}^{\mathbb{R}}\mathcal{C}^\infty(M)$ is a regular topological algebra [65].

As noted before, in the finitary context too, incidence Rota algebras'—ones taken to model finitary topological spaces, not qausets—Gel'fand duality and, in particular, the aforesaid method of Gel'fand spatialization was first applied in [129] and then further explored in [92, 93]. The basic idea there was to substitute the continuous spacetime poset-discretizations P_i of Sorkin in [114] by functional-like algebraic structures Ω_i, assign a topology to the latter, and then show how the original finitary poset topology may be identified with the latter. Thus, in complete analogy to the classical continuum case above, we used Gel'fand spatialization and:

1. Defined 'points' in the Ω_is as (kernels of finite dimensional) irreducible (Hilbert space) representations of them—that is, as elements of their primitive (maximal) spectra $\mathrm{Max}\Omega_i$.

2. Assigned a suitable topology on those primitive ideals.[316]

3. Identified the Rota topology on the primitive spectra of the Ω_is with the Sorkin topology of the P_is.

That the Ω_is are Gel'fand dual to the P_is is concisely encoded in the result quoted in section 4 that there is a (contravariant) functorial correspondence between the respective categories $\mathfrak{Z}$ and $\mathfrak{P}$.[317] In effect, this is precisely the correspondence that enables one to go from categorical (inverse, projective) limits in $\mathfrak{P}$ to categorical co- (direct, inductive) limits in (finsheaves of incidence algebras in) $\mathfrak{Z}$ mentioned above.[318] Furthermore, it was evident by the very structure of the Ω_is (as $\mathbb{Z}$-graded discrete differential manifolds) that, in the P_i-dual picture of incidence algebras, differential properties of the underlying space(time) could be studied, not just topological. In other words, in the finitary setting, Gel'fand duality revealed a differential structure that is encoded in the Ω_is which was 'masked' by the purely topological posets of Sorkin. With respect to the classical continuum paradigm of Gel'fand duality mentioned above, the analogy is clear:

> The P_is are the reticular analogues of M regarded solely as a $\mathcal{C}^0$-manifold, while the Ω_is as the reticular analogues of M regarded as a differential manifold [92, 93].

In fact, precisely due to this suggestive analogy it was intuited in [92, 93] that at the limit of infinite refinement of the locally finite coverings of the bounded region of X not only the topological, but also the differential structure of the continuum could be recovered. Heuristically speaking, the Ω_is' revealing of differential geometric attributes suggested to us that also 'change',[319] not only 'static' topological or 'spatial' relations, could be modelled algebraically and by finitary means.

Thus, as it was described in the previous section, in the sequel, Gel'fand duality associating incidence algebras (qausets) to locally finite posets modelling causets was first exploited in [86]

[316]This is the aforementioned 'non-standard' Rota topology, since it was argued that the Gel'fand (or the Zariski) topology on $\mathrm{Max}\Omega_i$ is trivial (*ie*, the discrete—Hausdorff or T_2—topology) [129, 92, 93].

[317]As also mentioned in footnote 160 in 4.3, the correspondence (construction) 'finitary posets'$\longrightarrow$'incidence algebras' is functorial precisely because the P_is are simplicial complexes [92, 93, 130].

[318]As also noted in footnote 160, precisely due to the functoriality of the correspondence (construction) 'finitary posets'$\longrightarrow$'incidence algebras', finsheaves in the sense of [87] exist.

[319]For any differential operator 'd' models change!

by using Sorkin's fundamental insight in [115] that it is more physical to think of a partial order as causality (*ie*, as a 'temporal' structure) than as topology (*ie*, as a 'spatial' structure). Furthermore, Sorkin's demand for a dynamical scenario for causets almost mandated to us the use of sheaf theory—that is, to organize the incidence algebras modelling qausets to sheaves of an appropriate, finitary kind [87]. Thus, curved finsheaves of incidence algebras were born as kinematical spaces for the dynamical variations of qausets out of blending this causal version of Gel'fand duality with the ideas, working philosophy and technical panoply of ADG [74, 75].

The bottom line of all this is that the semantic essence of Gel'fand duality—*ie*, to substitute the topology of the background 'space(time)' by the functions that live on 'it'—found its natural home in ADG, which, as we emphasized repeatedly above, similarly directs one to pay more attention on the objects (fields) that live on space(time) rather than on spacetime *per se*, independently of whether the latter is taken to be a reticular base topological space or a continuum. In fact, we may further hold that

> at a differential geometric, not just at a topological, level, ADG in some sense 'breaks' Gel'fand duality,[320] since it tells us that *the differential geometric structure (mechanism) comes directly from the (algebraic) objects that live (in the stalks of the algebra sheaves on) space(time), not from the base space(time) itself.*[321]

All in all, and from a causal perspective, Gel'fand duality, coupled to ADG, allowed us to 'differential geometrize' and, as a result, (dynamically) vary Sorkin *et al.*'s causets thus bring causet theory, which is a '*bottom-up*' approach to quantum gravity, closer to other '*top-down*' approaches, such as Ashtekar *et al.*'s.[322]

5.5.2 Projective limits of fcqv-Einstein equations: the power of differential triads

We conclude this subsection by presenting an inverse system $\overleftarrow{\mathcal{E}}$ of fcqv-Einstein equations like (124) which produces the generalized classical (*ie*, $\mathcal{C}^\infty$-`smooth`) vacuum Einstein equations for Lorentzian gravity at the categorical (projective) limit of infinite refinement or localization of the qausets. The discussion below shows just how powerful the basic ADG-theoretic notion of a differential triad is, since there is a hierarchy or 'tower' of projective/inductive systems of finitary

[320] Gel'fand duality understood here as a 'topological symmetry' between the underlying space(time) and the objects (functions) that dwell on it.

[321] Thus, when one is interested solely in the topological structure of the continuum M, the aforedescribed classical 'reconstruction result' of the manifold M from the algebra ${}^{\mathbb{R}}\mathcal{C}^\infty(M)$ shows precisely that the $\mathcal{C}^0$-topology of M can be recovered from ${}^{\mathbb{R}}\mathcal{C}^\infty(M)$ by Gel'fand spatialization, while the differential structure inherent in ${}^{\mathbb{R}}\mathcal{C}^\infty(M)$ is not essentially involved. Similarly, at the finitary level, we saw above how the Ω_i revealed a rich differential geometric structure that the purely topological finitary posets of Sorkin in [114] simply lacked. Of course, it must be noted here that since the spectrum of ${}^{\mathbb{R}}\mathcal{C}^\infty(M)$ can be identified (by Gel'fand duality) with M set-theoretically (*ie*, by a bijective map, which moreover is a homeomorphism) one can also automatically transfer from M to $\mathfrak{M}[{}^{\mathbb{R}}\mathcal{C}^\infty(M)]$ the classical differential (*ie*, $\mathcal{C}^\infty$-smooth) structure. But this is another issue. Notwithstanding (first author's hunch), there might be lurking here an appropriate '*representation theorem*' that would close the circle!

[322] This 'bottom-up' and 'top-down' distinction of the approaches to quantum gravity is borrowed from [30]. In relation to the three categories of approaches mentioned in the prologue, category 1 may be thought of as consisting of top-down approaches, while both categories 2 and 3 as consisting of bottom-up approaches.

structures which has at its basis $\overset{\rightleftarrows}{\mathcal{T}}:=\{\vec{\mathfrak{T}}_i\}$—the inverse system of fcq-differential triads (or its direct version $\overset{\rightleftarrows}{\mathcal{T}}$).

Anticipating some comments on singularities in the next section, we also discuss the intriguing result that the $\mathcal{C}^{\infty}$-`smooth` Einstein equations at the projective limit hold over a 'spacetime' that may be infested by singularities—in other words, *the gravitational law does not 'break down' at the latter* since, anyway, an fcqv-version of it appears to hold for every member of the system $\overleftarrow{\mathcal{E}}$ and the latter are structures reticular, 'singular' and quite remote from the featureless smooth continuum. On the contrary, singularities may be incorporated into (or absorbed by) the structure sheaf of the $\mathcal{C}^{\infty}$-`smooth` differential algebras so that the generalized differential geometric mechanism continues to hold over them and the theory still enables one to perform calculations in their presence[323] [77, 78, 73].

But let us present straight away the aforesaid hierarchy of projective/inductive families of finitary structures, commenting in particular on the projective system $\overleftarrow{\mathcal{E}}$ mentioned above. The diagram below as well as the discussion that follows it will also help us recapitulate and summarize certain facts about the plethora of inverse and direct systems we have encountered throughout the present paper.

$$\boxed{\textbf{The 11-storeys' tower of finitary inverse/direct systems}}$$

[323] In the same way that ADG enabled us earlier to 'see through' the fundamental discreteness of the base causets and write a perfectly legitimate *differential* (Einstein) equation over them, in spite of them.

$$
\begin{array}{c}
\boxed{\textbf{Level 7}: \text{ Inverse system } \overleftarrow{\mathcal{Z}} \text{ of `fully covariant}' \text{ fcqv} - \text{path integrals}}\\
\Big\uparrow\\
\boxed{\textbf{Level 6}: \text{ Inverse system } \overleftarrow{\mathfrak{E}} \text{ of fcqv} - \text{E} - \text{L} - \text{fields and their curvature spaces}}\\
\Big\uparrow\\
\boxed{\textbf{Level 5}: \text{ Inverse system } \overleftarrow{\mathcal{E}} \text{ of fcqv} - \text{Einstein equations}}\\
\Big\uparrow\\
\boxed{\textbf{Level 4}: \text{ Inverse system } \overleftarrow{\mathcal{M}} \text{ of (self} - \text{dual) fcqv} - \text{moduli spaces}}\\
\Big\uparrow\\
\boxed{\textbf{Level 3}: \text{ Inverse system } \overleftarrow{\mathcal{EH}} \text{ of (self} - \text{dual) fcqv} - \text{Einstein} - \text{Hilbert action functionals}}\\
\Big\uparrow\\
\boxed{\textbf{Level 2}: \text{ Inverse system } \overleftarrow{\mathfrak{A}} \text{ of affine spaces of (self} - \text{dual) fcqv} - \text{dynamos}}\\
\Big\uparrow\\
\boxed{\textbf{Level 1}: \text{ Inverse system } \overleftarrow{\mathcal{G}} \text{ of principal finsheaves and their (self} - \text{dual) fcqv} - \text{dynamos}}\\
\Big\uparrow\\
\boxed{\textbf{Level 0}: \text{ Inverse} - \text{direct system } \overset{\rightleftarrows}{\mathcal{T}} \text{ of fcq} - \text{differential triads}}\\
\Big\uparrow\\
\boxed{\textbf{Level } -\mathbf{1}: \text{ Inverse system } \overleftarrow{\mathcal{S}} \text{ of finsheaves of continuous functions}}\\
\Big\uparrow\\
\boxed{\textbf{Level } -\mathbf{2}: \text{ Direct system } \overrightarrow{\mathfrak{R}} \text{ of incidence Rota algebras or qausets}}\\
\Big\uparrow\\
\boxed{\textbf{Level } -\mathbf{3}: \text{ Inverse system } \overleftarrow{\mathcal{P}} \text{ of finitary substitutes or causets}}
\end{array}
\tag{150}
$$

Short stories about the eleven storeys

- **Levels −3 to −1:** The first three 'underground levels' can be thought of as assembling the fundamental one at level zero. Indeed, as explained in section 4, each member $\vec{P}_i$ of $\overleftarrow{\mathcal{P}}$ (now causally interpreted as a causet) comprises the base causal-topological space of each fcq-differential triad $\vec{\mathfrak{T}}_i$ in $\overset{\rightleftarrows}{\mathcal{T}}$ bearing the same finitarity index (level −3). Correspondingly (by Gel'fand duality), each member (qauset) $\vec{\Omega}_i$ of $\overrightarrow{\mathfrak{R}}$ comprises the reticular coordinate

algebras, the bimodules of differentials over them and the differential operators linking spaces of discrete differential forms of consecutive grade (level -2) that, when organized as finsheaves (level -1) over the base causets of level -3, yield the inverse-direct system $\overset{\rightleftarrows}{\mathfrak{T}}$ of fcq-differential triads of level 0.

- **Level 0:** This is the fundamental, 'ground level' of the theory in the sense that all the inverse systems at levels ≥ 1 have at their basis $\overset{\rightleftarrows}{\mathfrak{T}}$, as follows:

- **Level 1:** The inverse system $\overleftarrow{\mathcal{G}}$ of principal Lorentzian finsheaves $\vec{\mathcal{P}}_i^{\uparrow}$ of (reticular orthochronous spin-Lorentzian or causal symmetries of) qausets and their non-trivial (*ie*, non-flat, as well as self-dual) fcqv-dynamos $\vec{\mathcal{D}}_i^{(+)}$ can be obtained directly from $\overset{\rightleftarrows}{\mathfrak{T}}$ by (sheaf-theoretically) localizing or 'gauging' qausets in the stalks of the finsheaves in the corresponding (*ie*, of the same finitarity index) fcq-triads $\vec{\mathfrak{T}}_i \in \overset{\rightleftarrows}{\mathfrak{T}}$ [74].

- **Level 2:** The projective system $\overleftarrow{\mathfrak{A}}$ of affine spaces $\vec{\mathcal{A}}_i^{(+)}$ of (self-dual) fcqv-dynamos $\vec{\mathcal{D}}_i^{(+)}$ on the $\vec{\mathcal{P}}_i^{\uparrow}$s (or better, on the $\vec{\mathcal{E}}_i^{\uparrow}$s associated with the $\vec{\mathcal{P}}_i^{\uparrow}$s) can be obtained straightforwardly from $\overleftarrow{\mathcal{G}}$.

- **Level 3:** The inverse system $\overleftarrow{\mathcal{EH}}$ of (self-dual) fcqv-Einstein-Hilbert action functionals can be easily obtained from $\overleftarrow{\mathfrak{A}}$ if we recall from (125–126) the finitary version $\overrightarrow{\mathfrak{EH}}_i^{(+)}$ of the ADG-theoretic definition of the E-H action functional $\mathfrak{EH}$ in (65–66).

- **Level 4:** Similarly to $\overleftarrow{\mathcal{EH}}$, the inverse system $\overleftarrow{\mathcal{M}}$ of (self-dual) fcqv-moduli spaces $\vec{\mathcal{M}}_i^{(+)}$ in (120) can be obtained from the inverse system $\overleftarrow{\mathfrak{A}}$ member-wise, that is to say, by quotienting each $\vec{\mathcal{A}}_i^{(+)}$ in $\overleftarrow{\mathfrak{A}}$ by the automorphism group $\overrightarrow{\mathcal{A}ut_i\mathcal{E}}_i^{\uparrow}$ of the causon.

- **Level 5:** The projective system $\overleftarrow{\mathcal{E}}$ of fcqv-E-equations as in (124) is the main one we wish to discuss here. It can be readily obtained, again member-wise from $\overleftarrow{\mathcal{EH}}$, by varying each $\overrightarrow{\mathfrak{EH}}_i^{(+)}$ in the latter collection with respect to the (self-dual) fcqv-dynamo $\vec{\mathcal{D}}_i^{(+)}$ in each member of $\overleftarrow{\mathcal{G}}$, as in (the finitary version of) (67–70). The important thing to mention here is that the inverse, continuum, 'correspondence limit' [92, 74, 93, 75] of these fcqv-E-equations yields the 'generalized classical' vacuum Einstein equations for Lorentzian gravity on the $\mathcal{C}^\infty$-`smooth` spacetime manifold M which (*ie*, whose coordinate structure sheaf $\mathcal{C}_M^\infty$), *prima facie*, may have singularities, other general pathologies and anomalies of all sorts. We thus infer that, by ADG-theoretic means,

> *we are able to write the law of gravitation over a spacetime that may be teeming with singularities.*

In other words, and in characteristic contradistinction to the classical $\mathcal{C}^\infty$-manifold based general relativity, *the Einstein equations do not 'break down' near singularities, and the*

gravitational field does not stumble or 'blow-up' at them. Rather, it evades them, it 'engulfs' or 'incorporates' them them,[324] *it holds over them and, as a result, we are able to calculate over them* [73].[325] Indeed, it has been shown [70] that with the help of ADG one can write the gravitational vacuum Einstein equations over the most pathological, especially when viewed from the $\mathcal{C}^\infty$-perspective, space(time) M—one whose structure sheaf $\mathbf{A}_M$ consists of Rosinger's differential algebras of generalized functions which, as noted earlier, have singularities on arbitrary closed nowhere dense subsets of M or even, more generally, on *arbitrary sets with dense complements* [77, 78, 97].

- $\boxed{\textbf{Levels 6 and 7:}}$ We will briefly comment on the last two remaining projective systems, $\overleftarrow{\mathfrak{E}}$ and $\overleftarrow{\mathcal{Z}}$. The first is supposed to consist of (self-dual) fcqv-E-L fields $(\vec{\mathcal{E}}_i^{\uparrow}, \vec{\mathcal{D}}_i^{(+)})$ and their corresponding curvature space pentads $(\vec{\mathbf{A}}_i, \vec{\partial}_i \equiv \vec{d}_i^0, \vec{\mathbf{\Omega}}_i^1, \vec{\mathrm{d}}_i \equiv \vec{d}_i^1, \vec{\mathbf{\Omega}}_i^2)$.[326] In line with footnote 58, we suppose that these fcqv-curvature spaces and the fcqv-E-spaces $\vec{P}_i$ supporting them are the 'solution spaces' of the corresponding equations in $\overleftarrow{\mathcal{E}}$. At the same time, it must be noted that this 'gedanken supposition'—that is, that curvature spaces refer directly to solutions of the fcqv-E-equations—is made to further emphasize the point made at the previous level, namely, that: in case one obtains an fcqv-E-L-field $(\vec{\mathcal{E}}_i^{\uparrow}, \vec{\mathcal{D}}_i^{(+)})$ (and therefore its curvature $\vec{\mathcal{R}}_i^{(+)}(\vec{\mathcal{D}}_i^{(+)})$) that is a solution of (124), then the projective, $\mathcal{C}^\infty$-continuum limit of these solutions may be infested by singularities ,*but still be a legitimate solution of (ie, satisfy) the* ***smooth*** *vacuum Einstein equations and the singularities did not in any way 'inhibit' the physical law or our calculations with it.*[327] We can summarize all this with the following statement quoted almost *verbatim* from [73]:

  > *A physical law cannot be dependent on, let alone be restricted by, singularities.*[328]

  This may be perceived as further support to Einstein's doubts in [38]:

  > "*It does not seem reasonable to me to introduce into a continuum theory points (or lines etc.) for which the field equations do not hold.*"[329]

  As for the inverse system $\overleftarrow{\mathcal{Z}}$ whose members are heuristic covariant fcqv-path integrals $\vec{\mathcal{Z}}_i$ *à la* (128), our comments for its projective continuum limit must wait for results from the ADG-theoretic treatment of functional integration in gauge theories currently under

[324] This is so because the observable the gravitational field strength is an $\mathbf{A}$-morphism (*ie*, it respects the generalized arithmetics in $\mathbf{A}$), and the generalized coordinate algebras in the structure sheaf may include arbitrarily potent singularities.

[325] We are going to comment further on this in the next section.

[326] Which in turn, as noted in 5.1, makes the base causet $\vec{P}_i$ an *fcqv-E-space*.

[327] These solutions being, in fact, the results of our calculations in the presence of the singularities incorporated in our own arithmetics $\mathbf{A}$!

[328] Equivalently, *Nature has no singularities* (see next section).

[329] And Einstein's doubts are remarkable indeed if one considers that they are expressed in the context of classical field theory on a $\mathcal{C}^\infty$-smooth spacetime manifold M with the unavoidable singularities that infest its coordinate structure sheaf $\mathcal{C}^\infty_M$.

development in [71]. Our hunch is that, if the fcqv-E-H-action $\overrightarrow{\mathfrak{EH}}_i$ involved in the integrand of $\vec{\mathcal{Z}}_i$ is taken to be a functional of the self-dual fcqv-dynamo $\vec{\mathcal{D}}_i^+$ (write: $\overrightarrow{\mathfrak{EH}}_i^+$ and, *in extenso*, $\vec{\mathcal{Z}}_i^+$), the continuum limit should yield the generalized $\mathcal{C}^\infty$-version of the $\mathcal{C}^\infty$-path integral involving the exponential of the `smooth` analogue of the smooth Asthekar action S_{ash} in (129).

6 Epilogue: the Wider Physical Significance of ADG

In this concluding section we would like to discuss the wider physical implications of our work here and of ADG in general. We concentrate on two aspects: on the one hand, how ADG may potentially help us evade the notorious $\mathcal{C}^\infty$-singularities, thus we prepare the ground for a paper that is currently in preparation [76], and on the other, how ADG points to a 'relativized' notion of differentiability.

6.1 Towards Evading $\mathcal{C}^\infty$-Smooth Singularities

We would like to commence our brief comments on smooth singularities, anticipating a more elaborate treatment in [76], with the following two quotations of Isham:

> *"...A major conceptual problem of quantum gravity is...the extent to which classical geometrical concepts can, or should, be maintained in the quantum theory..."* [53]

[principally because][330]

> *"...The classical theory of general relativity is notorious for the existence of unavoidable spacetime singularities..."* [54]

which are completely analogous to the two quotations in the beginning of the paper. For instance, one could combine Feynman's and Isham's words in the following way:

> *one cannot apply classical differential geometry in quantum gravity, because one gets infinities and other difficulties.*

Indeed, it is generally accepted that if one wishes to approach the problem of quantum gravity by assuming up-front that spacetime is (modelled after) a $\mathcal{C}^\infty$-smooth manifold,[331] one's theory would be plagued by singularities well before quantization proper becomes an issue—that is, long before one had to address the problem of actually quantizing the classical theory. In other words, the problem of singularities is already a problem of the classical theory of gravity that appears to halt the program of quantizing general relativity already at stage zero. Even if one turned a blind eye to the singularities of the classical theory and proceeded to tackle quantum general

[330] Our addition in order to link the two together.
[331] Such an approach would belong to the 'calculus conservative' category 1 mentioned in the prologue.

relativity as another quantum field theory based again on the classical spacetime continuum, one would soon encounter gravitational infinities that, although milder and less robust than the singularities of the classical theory, they are strikingly *non-renormalizable*[332] in contradistinction to the infinities of the quantum field theories of gauge matter which are perturbatively finite. Altogether, it is the $\mathcal{C}^\infty$-manifold M (with its structure ring $\mathcal{C}^\infty(M)$ of infinitely differentiable functions) employed by the usual differential geometry supporting both the classical and the quantum general relativity which is responsible for the latter's "*unavoidable spacetime singularities*" and unremovable infinities, and which makes classical (differential) geometric concepts and constructions appear to be *prima facie* inapplicable in the quantum deep.

On the other hand, the word 'unavoidable' in Isham's quotation [54] above calls for further discussion, because it goes against the grain of the very basic didactics of ADG *vis-à-vis* singularities [78, 70, 73]. It now appears clear to us that the singularities of general relativity come from assuming up-front $\mathcal{C}^\infty_M$ as the structure sheaf $\mathbf{A}$ of 'coefficients' over which one applies the classical differential geometric constructions to classical gravity. Since the differential pathologies are due to $\mathcal{C}^\infty(M)$, the whole enterprize of applying (differential) geometric concepts to classical and, *in extenso*, to quantum gravity, seems to be doomed from the start. On the other hand, ADG has taught us precisely that *singularities are indeed avoidable if one uses a different and more 'suitable' to the physical problem at issue structure arithmetics* $\mathbf{A}$ *than* $\mathcal{C}^\infty_M$ [68, 78, 70, 73]. Moreover, ADG has time and again shown that the 'intrinsic mechanism' of the classical differential geometry ($\mathbf{A}_X \equiv \mathcal{C}^\infty_X$) can be carried over, intact, to a generalized differential geometric setting afforded by a general structure sheaf $\mathbf{A}$ very different from $\mathcal{C}^\infty_M$ [77, 78, 70, 75]. Since $\mathbf{A}$ can be taken to include arbitrary singularities, even of the most extreme and classically unmanageable sort [77, 78, 97], it follows that *the said differential mechanism is genuinely independent of singularities.* That is to say,

> *not only we can avoid singularities ADG-theoretically, but we can actually absorb or 'engulf' them into* $\mathbf{A}$ (provided of course these algebras are 'appropriate' or 'suitable' for serving as the structure arithmetics of the abstract differential geometry that has been developed[333]) *and, as a result, calculate or perform our (differential geometric) constructions over them, in spite of their presence which thus becomes unproblematic* [73].[334]

These remarks bring to mind Einstein's 'apologetic confession':[335]

[332] Essentially due to the dimensionfulness of Newton's gravitational constant.

[333] That is to say, they can provide us with the basis for defining differentials, connections, vectors, forms and higher order $\otimes_{\mathbf{A}}$-tensors, as well as the rest of the 'differential geometric apparatus' in much the same way that $\mathcal{C}^\infty_M$ does, supported by the smooth manifold M, in the classical theory.

[334] In a straightforward way, ADG shows that singularities can be integrated into the structure algebra sheaf $\mathbf{A}$ of our own 'generalized measurements', 'arithmetics' or 'coefficients', thus they should never be regarded as problems of Physis. In other words, *Nature has no singularities, rather, it is our own models of Her that are of limited applicability and validity* (*eg*, in the classical case this pertains to the $\mathcal{C}^\infty$-smooth manifold model M for spacetime, the structure sheaf $\mathbf{A} \equiv \mathcal{C}^\infty_M$ that it supports, and the $\mathcal{C}^\infty$-singularities that the latter hosts).

[335] Which we encountered earlier in 5.4.1. We too apologize for displaying this quotation twice, but we find it very suggestive and relevant to one of the main points that we would like to make in the present paper, namely,

> *"...Adhering to the continuum originates with me not in a prejudice, but arises out of the fact that I have been unable to think up anything organic to take its place..."* [36],

in the sense that Einstein's commitment to the continuum and, in effect, to the classical differential geometry supporting his theory of gravitation, would not have been as strong or as faithful[336] had there been an alternative (mathematical) scheme—perhaps one of a strong algebraic character if one considers his life-long quest (in view of quantum theory and the pathologies of the continuum) *for an entirely algebraic description of reality*[337]—that worked as well as the $\mathcal{C}^\infty$-differential geometry, yet, unlike the latter, was more algebraic, not dependent on a dynamically inactive spacetime continuum and, perhaps more importantly, it was not assailed by singularities, infinities and other 'differential geometric diseases' coming from the *a priori* assumption of the smooth background manifold.[338] We contend that ADG is a candidate for the algebraic theory that Einstein had envisioned, for, as we saw here and in a series of papers [77, 78, 74, 75], one can carry out all the differential geometric constructions that are of use in the usual differential geometry supporting general relativity with the help of suitable vector and algebra sheaves over arbitrary base spaces—even over ones that are extremely singular and reticular when viewed from the perspective of the smooth continuum. Thus, in effect,

> *according to ADG, (the intrinsic or inherent mechanism of) differential geometry has nothing to do with the background space so that, in particular, it is not affected by the singularities of the manifold* [73].

For the sake of completeness, we would like to bring to the attention of the reader two examples from the physics literature, one old the other new, of theories that evade singularities in a way that accords with the general spirit of ADG described above.

- **Evading the exterior Schwarzschild singularity (old).** The paradigm that illustrates best how a change in the coordinate structure functions or generalized arithmetics $\mathbf{A}$ may effectively resolve a singularity is Finkelstein's early work on the gravitational field of a point particle [41]. It was well known back then that the Schwarzschild solution of the Einstein equations for the gravitational field of a point mass m had two singularities: an exterior one, at distance (radius) $r = 2m$ from m, and an interior right at the point mass ($r = 0$). What Finkelstein was able to show is that by an appropriate change of coordinates[339]—the so-called Eddington-Finkelstein frame, the exterior singularity is 'transformed

that *if Einstein had a way* (*ie*, a theory and a working method) *of doing field theory—and differential geometry in general—independently of the pathological and unphysical spacetime continuum, and, moreover, by finitistic-algebraic means (in view of the quantum paradigm), he would readily abandon the* $\mathcal{C}^\infty$*-smooth manifold* (see more remarks shortly). We claim that ADG, especially in its finitary guise here, is such a theory.

[336] Quite remarkably though, considering that general relativity enjoyed numerous successes and was experimentally confirmed during Einstein's life.

[337] See the three quotations in 5.1.1.

[338] In these terms we may understand the epithet '*organic*' above.

[339] However, always in the context of a smooth spacetime manifold M (*ie*, still with the new coordinate functions being members of $\mathbf{A} \equiv \mathcal{C}^\infty_M$).

away' revealing that the Schwarzschild spacetime acts as a unidirectional, 'semi-permeable', time-asymmetric membrane allowing the outward propagation of particles and forbidding the inward flux of antiparticles. For this, the $r = 2m$ singularity was coined '*coordinate singularity*' and was regarded as being only a *'virtual' anomaly*—merely an indication that we had laid down inappropriate coordinates to chart the gravitational spacetime manifold.

On the other hand, it was also realized that the interior singularity could not be gotten rid of by a similar coordinate change,[340] thus it was held as being a *'real'* or *'true' singularity*—an alarming indication that general relativity is out of its depth when trying to calculate the gravitational field right on its point source. Thus, ever since Finkelstein's result, it has been hoped that only a genuine quantum theory of gravity will be able to deal with the gravitational field right at its source much in the same way that the quantum theoresis of electrodynamics (QED) managed, even with just the theoretically rather *ad hoc* method of 'subtracting infinities' (renormalization),[341] to do meaningful physics about the photon radiation field at its source—the electron.

According to this rationale however, notwithstanding the perturbative non-renormalizability of gravity due to the dimensionality of Newton's constant, it has become obvious that physicists have devotedly committed themselves so far to viewing the spacetime point manifold as something physically 'real' in the sense that any of its points is regarded as potentially being the host of a non-circumventable by $\mathcal{C}^\infty$-means singularity for a physically important smooth field. That is, instead of reading Finkelstein's result in a positive way, as for instance in the following manner *à la* ADG,

> *when encountering any singularity, in order to 'resolve' it and be able to cope with (ie, calculate over) it, one must look for an 'appropriate' structure algebra of coordinates that incorporates or 'engulfs' it* [73] *and then one has to give a cogent physical interpretation of the new picture*,[342]

physicists try instead to retain as much as they can (admittedly, by ingenious methods at times) the differential spacetime manifold M, its structure coordinates $\mathbf{A} \equiv \mathcal{C}^\infty_M$ and its structure symmetries $\mathcal{G} \equiv \mathrm{Diff}(M)$ as if they were physically real, and at the same time quite falsely infer that the mechanism of (classical) differential geometry does not apply over singularities and, *in extenso*, in the quantum deep.[343] All in all, it is as if:

[340]Again though, still by remaining within the $\mathcal{C}^\infty$-smooth manifold model.

[341]It is well known, for instance, that Dirac expressed many times his dissatisfaction about the renormalization program with its mathematically not well founded and aesthetically unpleasing recipes: "*Sensible mathematics involves neglecting a quantity when it turns out to be small—not neglecting it just because it is infinitely great and you do not want it*" [29].

[342]The word 'appropriate' meaning here in the manner of ADG: a (differential) algebra of coordinates that integrates the singularity (as a generalized coefficient) yet it is still able to provide us with the basic differential mechanism we need in order to set up the relevant dynamical equations over it and calculate with them.

[343]Such an attitude was coined in [75] '*$\mathcal{C}^\infty$-smooth manifold conservative*' and it is the spirit underlying category 1 of approaches to quantum gravity mentioned in the prologue. For instance, physicists try to isolate and surgically cut-out of the spacetime manifold the offensive singular points, thus continue the usual $\mathcal{C}^\infty$-differential geometric

1. The smooth spacetime manifold is a physically real substance to be retained by all means.
2. The $\mathcal{C}^\infty$-singularities are also physically real as they are Nature's (*ie*, the spacetime manifold's) own diseases—they are real physical problems, 'intrinsic' pathologies of Nature (spacetime).
3. The (classical) differential calculus and the dynamical laws (*eg*, the Einstein equations) supported by it break down at a singularity.
4. In order to retain the spacetime manifold so that one can continue doing calculus (*ie*, apply the usual differential geometric ideas and techniques to physical situations—as it were, 'continue the validity of physical laws' and, in fact, *calcul*ate!), singularities must be isolated and then somehow removed or 'surgically excised' from the manifold, leaving back an effective spacetime manifold free of pathologies.

At the same time, a natural follow-up of this line of thought is the following basic hunch shared nowadays by almost all the workers in the field of quantum gravity (string theorists aside) looking for alternatives to the spacetime continuum of macroscopic physics,[344]

> *at strong gravitational fields near singularities, or at Planck distances, the conventional image of spacetime as a smooth continuum breaks down and should somehow give way to something 'discrete', 'reticular', 'inherently cut-off', and this should be accompanied by a radical modification of the classical differential geometry used to describe classical, 'low energy' Einstein gravity on M. At the core of this philosophy hibernates the idea that the notion of spacetime—be it discrete or continuous—must be retained at any cost, and that our methods of calculation must be modified accordingly, as if all our constructions must be tailor-cut to suit (or better, derive from) a pre-existent background geometrical space(time).*[345]

- **Passing through the initial singularity by ekpyrosis (new).** Together with the interior Schwarzschild singularity, there is another one, perhaps even more famous, which is a direct consequence of Einstein's general theory of relativity, namely, the *initial Big*

practices in the remaining 'effective manifold'. (In a sense, they 'artificially' remove, by hand and force as it were, the "*points, lines etc for which the field equations do not hold*", as we read in Einstein's quotation at the end of the last section.) Current physics regards singularities as an incurable disease of differential geometry. In contradistinction, ADG maintains that they are unmanageable indeed by $\mathcal{C}^\infty$-means, but also, more importantly, that the (algebraic in nature) differential mechanism is not affected by them, so that one should be able to continue 'calculating' in their presence.

[344] To name a few alternative schemes to the spacetime continuum and to the classical theory of gravity that it supports: simplicial (Regge) gravity, spin-networks, causet theory *etc*.

[345] In spite of Einstein's serious doubts about the physical meaningfulness of the concepts of space and time mentioned earlier. Even more remarkably, in 4.2.2 we mentioned how Isham has contemplated changing drastically the standard quantum theory itself in order to suit non-continua spacetime backgrounds, such as causal sets for example.

Bang singularity marking the beginning of an expanding Universe in the most successful of modern cosmological models. The initial singularity, like the aforementioned interior Schwarzschild one, is regarded as a fundamental, 'true' spacetime singularity and physics during the Planck epoch (0—$10^{-42}s$) is anticipated to be described consistently by the ever elusive quantum theory of gravity. However, recently, in the context of the string, membrane and M-theory approach to quantum gravity, Khoury *et al.* have proposed a scenario according to which one can actually evade the initial singularity—as it were, do meaningful pre-Big Bang era physics [58, 59, 120]. Without going into any technical details, we just note that their proposal basically involves a (coordinate) field transformation,[346] completely analogous to Finkelstein's frame change in [41],[347] which enables one to go through the initial singularity as if it was a diaphanous membrane. Thus, even the most robust and least doubted singularity of all, the Big Bang, has been shown (again, simply by the use of $\mathcal{C}^\infty$-means!) to be no problem, no pathology of Nature at all, and that a rich physics is to be discovered even for the period 'before time began'.

6.2 The Relativity of Differentiability

In connection with our brief remarks on $\mathcal{C}^\infty$-singularities above, we wish to close the present paper with further remarks on the opening two quotations of Feynman and Isham. In particular, in line with the discussion of 'gravity as a gauge theory' in section 3, we would like to emphasize that,

1. while we share Feynman's scepticism about the metric-formulation of general relativity[348] and his hunch that there is a fundamental gauge invariance lurking there,

2. we do not share his apparently 'negative' stance towards differential geometry. Of course, his position is understandable to the extent that he is referring to (and he is actually referring to!) the usual calculus on $\mathcal{C}^\infty$-manifolds, but this is precisely the point of ADG: *one should not question the 'differential mechanism' per se when encountering singularities, infinities and other pathologies in classical differential geometry.* For, loosely speaking, '*the mechanism is fine*', *as it works, that is, as one can actually do differential geometry in principle over any space, no matter how singular.* Rather, *one should question the $\mathcal{C}^\infty$-smooth manifold M itself whose only operative role in the said 'differential mechanism' is*

[346] Still assuming however $\mathcal{C}^\infty$-smoothness for the various fields involved (*ie*, $\mathbf{A} \equiv \mathcal{C}^\infty_X$ in our language; where X is a higher-dimensional differential manifold, *eg*, a Riemann hypersurface).

[347] Neil Turok in private communication [123].

[348] After all, *the metric, as well as the space hosting it, are our own ascriptions to Physis; they are not Nature's own* (recall Einstein's quotation [36] in 5.1.1). ADG emphasizes that the $\mathbf{A}$-metric ρ, as the term suggests, is crucially dependent on our own measurements or 'generalized arithmetics' in $\mathbf{A}$, so that, like the singularities of the previous subsection, it is not Nature's own property: *we ascribe it to Her!* (See footnote 17.) This is in line with quantum theory's basic algebraico-operationalist philosophy (and goes against the Platonic realist ideal of classical physics) according to which, *quantum systems do not possess physical properties of their own, that is, independently of our acts of observing them. These acts, in turn, can be suitably organized into algebras of physical operations, generalized 'measurements' so to speak, on the quantum system.*

to provide us with the algebras (by no means unique or 'preferred' in any sense[349]*)* $\mathcal{C}^\infty(M)$ *of infinitely differentiable functions (and the classical differential geometric mechanism supported by them)* which, in turn, *are the very hosts of the aforementioned singularities and the other 'classical differential geometric diseases'*.

Since Feynman's stance appears to accord with Isham's,[350] our reply to the latter is similar; expressed somewhat differently,

3. we seem to be misled by the classical theory—the $\mathcal{C}^\infty$-differential geometry—into thinking that the various 'differential geometric pathologies' are faults and shortcomings of the differential mechanism, thus also infer that *differential geometry does not apply in the quantum deep*. As noted earlier, it is perhaps habit or long-time familiarity with smooth manifolds and their numerous successful applications to physics, including general relativity and the quantum field theories of matter, that makes us think so,[351] for ADG has shown us that *the differential mechanism still applies effectively over any space—even over ones that are much more singular (in a very straightforward, but technical, sense)* [77, 78, 97], *or even over ones that are manifestly discontinuous and more quantal* [74, 75], *than the 'featureless' differential manifold*. On the other hand, ADG has also shown us that the 'differential diseases' are exactly due to our assuming up-front a differential manifold background space to support our differential geometric constructions, thus agreeing in that sense with Feynman and Isham. However, in contradistinction to them,

   > *in view of ADG, one does not need the differential manifold in order to differentiate.*

   All in all, ADG suggests that

   > *to heal the differential pathologies, one must first kick the* $\mathcal{C}^\infty$*-smooth manifold habit.*

   Thus, continuing the 'sloganeering' with which we concluded [75][352] and expressed slogans 1–3 in the present paper, we may distill the remarks above to the following '*relativity of differentiability*' principle:

[349] See the *principle of relativity of differentiability* to follow shortly.

[350] See the two quotations opening the paper.

[351] See quotation of Einstein concluding the paper below. At this point, to give an indication of this attitude—*ie*, of the persistent, almost 'religious' adherence of some physicists to the spacetime manifold—we may recall Hawking's opening words in [50] where he discusses singularities in general relativity *vis-à-vis* quantum gravity: "*...Although there have been suggestions that spacetime may have a discrete structure, I see no reason to abandon the continuum theories that have been so successful. General relativity is a beautiful theory that agrees with every observation that has been made. It may require modifications on the Planck scale, but I don't think that will affect many of the predictions that can be obtained from it...*" This appears to be the *manifold-conservative* stance against singularities and quantum gravity *par excellence*.

[352] Especially, see slogan 2 there.

4. *The differential spacetime manifold by no means sets a preferred (ie, unique) frame (ie, model) for differentiating physical quantities. Differential equations, modelling physical laws that obey the generalized principle of locality,*[353] *can be also set up independently of the $\mathcal{C}^\infty$-smooth manifold—in fact, as we saw in this paper, regardless of any background (base) space(time).* Since we have repeatedly argued and witnessed in this paper that *differentiability derives from the stalk* (*ie*, from the algebraic objects dwelling in the relevant sheaves) and *not from the underlying space(time)*, we may say that the 'absolute' and fixed differentiability of the smooth spacetime manifold, which for Einstein represented the last relic of an inert, 'dynamically indifferent' ether-like substance [32, 34] that "*acts, but is not acted upon*" [38],[354] '*relativized*' with respect to the algebraic objects that live on whatever 'spacetime'[355] we have used as a base space 'scaffolding' to localize sheaf-theoretically those physically significant algebraic objects. We may figuratively refer to the abstract algebraico-sheaf-theoretic differentiability properties (of the system 'quantum spacetime'—or better, of the very dynamical quanta in which that 'spacetime' is inherent) as '*differentiables*', in analogy to the standard algebraically represented 'observables' or even the 'beables' of the usual (material) quantum physical systems. Thus, to wrap things up,

> *'Differentiables' are properties of (ie, derive from) the algebraic structure of the objects (sections of algebra sheaves) that live on 'space(time)', not from 'space(time)' itself which, especially in its classical $\mathcal{C}^\infty$-smooth manifold guise, is doubtful whether it has any physical significance at all* [53, 54, 92, 74, 21, 93,

[353]Which maintains that physical laws should be modelled after differential equations that depict the cause-and-effect nexus between 'infinitesimally' or 'smoothly separated' ('$\mathcal{C}^\infty$-contiguous') events—arguably what one understands by 'differential locality' (*ie*, local causality in the $\mathcal{C}^\infty$-smooth spacetime manifold) [74, 93].

[354]More precisely, Einstein's doubts about the physical reality of the absolute, dynamically passive spacetime continuum of the (special) theory of relativity were expressed in [38] (p. 55) as follows: "*...In this latter statement* [*ie*, that from the standpoint of special relativity *continuum spatii et temporis est absolutum*] *absolutum means not only 'physically real', but also 'independent of its physical properties, having a physical effect, but not itself influenced by physical conditions'...*" Indeed so, in the special theory of relativity the metrical properties of the spacetime continuum were not relativized, so that the metric was not regarded as a dynamical variable. The general theory of relativity viewed the metric—'the field of locality' (local causality or local chronology)—as a dynamical variable and effectively evaded the aforesaid '*temporis est absolutum*', but it must again be emphasized here that general relativity in a sense came short of fully relativizing (*ie*, regarding as dynamical variables) the whole panoply of structures (or 'properties' in Einstein's words above) that the spacetime continuum comes equipped with. For instance, the continuum's structures which are arguably 'deeper' than the metrical, such as the topological and the differential, are simply left absolute, non-relativized (non-dynamical), 'fixed by the theorist once and forever as the differential manifold background'. As noted repeatedly earlier and in previous works [92, 93, 74, 75], in a genuinely (fully) quantum theoresis of spacetime structure and dynamics even the topological and the differential structures are expected to be subjected to relativization and dynamical variability—thus *become 'observables', 'in principle measurable' dynamical entities*. For it has been extensively argued that *the common denominator of both relativity ('relativization') and the quantum ('quantization') is dynamics ('dynamical variation')* [45]. So that "*all is quantum*" (see footnote 3) means essentially that "*all is dynamical*". But then, if everything is in constant flux in the quantum deep, *whence space?*, and, *mutatis mutandis*, *whence time?* Totally, *is there any spacetime at all?*, and even more doubtfully, *whence the spacetime manifold?*.

[355]The inverted commas over 'spacetime' remind one of the physically dubious (especially at Planck scale) significance of this concept.

75, 73, 55].

However, since we have repeatedly quoted above Einstein's doubts about the smooth geometric spacetime continuum *vis-à-vis* singularities and the quantum, we would like to end the paper with another telling quotation of his which sensitizes us to the fact that successful, therefore *a priori* assumed and habitually or uncritically applied, theoretical concepts and mathematical structures,[356] can exercize so much power on us that they often mask their true origin and pragmatic usefulness—*ie*, that *they simply are our own theoretical constructs of limited applicability and validity*—and mislead us into thinking that they are 'unavoidable necessities' and, what's worse, Nature's own traits:

> "*...Concepts which have proved useful for ordering things easily assume so great an authority over us, that we forget their terrestrial origin and accept them as unalterable facts. They then become labelled as 'conceptual necessities', 'a priori situations', etc. The road of scientific progress is frequently blocked for long periods by such errors. It is therefore not just an idle game to exercise our ability to analyse familiar concepts, and to demonstrate the conditions on which their justification and usefulness depend, and the way in which these developed, little by little...*" (1916) [31]

Acknowledgments

The second author (IR) is indebted to Chris Isham for numerous discussions on quantum gravity, but more importantly, for his unceasing moral encouragement and support. He would also like to acknowledge helpful exchanges on finitary posets, simplicial complexes and causal sets with Raquel Garcia and Yousef Ghazi-Tabatabai, and to express his thanks to the European Union for funding this work via a Marie Curie postdoctoral research fellowship held at Imperial College.

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[356] Like, in our case, the $\mathcal{C}^\infty$-smooth spacetime manifold and its anomalies. As we also mentioned in 4.2.2, in [21, 55] too the dangers of assuming *a priori* the spacetime continuum in our excursions into the quantum deep are explicitly pronounced.

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